E. Wentworth

Corrections for new edition

ELEMENTS OF ALGEBRA.

BY

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PREFACE.

THE single aim in writing this volume has been to make an Algebra which the beginner would read with increasing interest, intelligence, and power. The fact has been kept constantly in mind that, to accomplish this object, the several parts must be presented so distinctly that the pupil will be led to feel that he is *mastering* the subject. Originality in a text-book of this kind is not to be expected or desired, and any claim to usefulness must be based upon the method of treatment and upon the number and character of the examples. About four thousand examples have been selected, arranged, and tested in the recitation-room, and any found too difficult have been excluded from the book. The idea has been to furnish a great number of examples for practice, but to exclude complicated problems that consume time and energy to little or no purpose.

In expressing the definitions, particular regard has been paid to brevity and perspicuity. The rules have been deduced from processes immediately preceding, and have been written, not to be committed to memory, but to furnish aids to the student in framing for himself intelligent statements of his methods. Each principle has been fully illustrated, and a sufficient number of problems has been given to fix it firmly in the pupil's mind before he proceeds to another. Many examples have been worked out, in order to exhibit the best methods of dealing with different classes of problems and the best arrangement of the work; and such aid has been given in the statement of problems as experience has shown

to be necessary for the attainment of the best results. General demonstrations have been avoided whenever a particular illustration would serve the purpose, and the application of the principle to similar cases was obvious. The reason for this course is, that the pupil must become familiar with the separate steps from particular examples, before he is able to follow them in a general demonstration, and to understand their logical connection.

It is presumed that pupils will have a fair acquaintance with Arithmetic before beginning the study of Algebra; and that sufficient time will be afforded to learn the *language* of Algebra, and to settle the principles on which the ordinary processes of Algebra are conducted, before attacking the harder parts of the book. "Make haste slowly" should be the watchword for the early chapters.

It has been found by actual trial that a class can accomplish the whole work of this Algebra in a school year, with one recitation a day; and that the student will not find it so difficult as to discourage him, nor yet so easy as to deprive him of the rewards of patient and successful labor. At least one-fourth of the year is required to reach the chapter on Fractions; but, if the first hundred pages are thoroughly mastered, rapid and satisfactory progress will be made in the rest of the book.

Particular attention should be paid to the chapter on Factoring; for a thorough knowledge of this subject is requisite to success in common algebraic work.

The materials for this Algebra have been obtained from English, German, and French sources. To avoid trespassing upon the works of recent American authors, no American text-book has been consulted.

The author returns his sincere thanks for assistance to Rev. Dr. Thomas Hill; to Professors Samuel Hart of Hartford, Ct.; C. H. Judson of Greenville, S.C.; O. S. Westcott of Racine, Wis.; G. B. Halsted of Princeton, N.J.; M. W. Humphreys of Nashville, Tenn.; W. LeConte Stevens of New York, N.Y.; G. W. Bailey of New

York, N.Y.; Robert A. Benton, of Concord, N.H.; and to Dr. D. F. Wells of Exeter. He has also the pleasure of expressing his obligations to Messrs. J. S. Cushing and F. E. Bartley, to whose superior taste and judgment the typographical excellence of this book is due.

Answers to the problems are bound separately in paper covers, and will be furnished free to pupils when *teachers* apply to the publishers for them.

Any corrections or suggestions relating to the work will be thankfully received.

G. A. WENTWORTH.

PHILLIPS EXETER ACADEMY,
May, 1881.

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ELEMENTS OF ALGEBRA.

CHAPTER I.

QUANTITY AND NUMBER.

1. WHATEVER may be regarded as being made up of parts like the whole is called a **Quantity.**

2. To **measure** a quantity of any kind is to find how many times it contains another *known quantity of the same kind.*

3. A *known quantity* which is adopted as a standard for measuring quantities of the same kind is called a **Unit.** Thus, the foot, the pound, the dollar, the day, are units for measuring distance, weight, money, time.

4. A **Number** arises from the repetitions of the unit of measure, and shows *how many times* the unit is contained in the quantity measured.

5. When a quantity is measured, the result obtained is expressed by prefixing to the *name* of the unit the *number* which shows how many times the unit is contained in the quantity measured; and the two combined denote a quantity expressed in units. Thus, 7 feet, 8 pounds, 9 dollars, 10 days, are quantities expressed in their respective units.

When a question about a quantity includes the unit, the answer is a *number;* when it does not include the unit, the answer is a *quantity.* Thus, if a man who has fifteen bushels of wheat be asked *how many bushels* of wheat he has, the answer is the *number*, fifteen; if he be asked *how much* wheat he has, the answer is the *quantity*, fifteen bushels.

A number answers the question, How many? a quantity, the question, How much?

Numbers.

6. The symbols which **Arithmetic** employs to represent numbers are the figures 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The *natural* series of numbers begins with 0; each succeeding number is obtained by adding one to the preceding number, and the series is infinite.

7. Besides figures, the chief symbols used in Arithmetic are:

$+$ (read, plus), the sign of addition.
$-$ (read, minus), the sign of subtraction.
$\times$ (read, multiplied by), the sign of multiplication.
$\div$ (read, divided by), the sign of division.
$=$ (read, is equal to), the sign of equality.

Exercise. — Read:

$$7+12=19. \qquad 8+3-5=20-15+1.$$
$$9-4=5. \qquad 24+6=10\times 3.$$
$$6\times 4=24. \qquad 14-7+5=6\times 2.$$
$$48\div 3=16. \qquad 9\times 5=180\div 4.$$

8. Any figure, or combination of figures, as 7, 28, 346, has one, and only one, value. That is, figures represent

particular numbers. But numbers possess many *general properties*, which are true, not only of a particular number, but of all numbers.

Thus, the sum of 12 and 8 is 20, and the difference between 12 and 8 is 4. Their sum added to their difference is 24, which is twice the greater number. Their difference taken from their sum is 16, which is twice the smaller number.

9. As this is true of any two numbers, we have this general property: *The sum of two numbers added to their difference is twice the greater number; the difference of two numbers taken from their sum is twice the smaller number.* Or,

1. (greater number + smaller number) + (greater number − smaller number) = twice greater number.

2. (greater number + smaller number) − (greater number − smaller number) = twice smaller number.

But these statements may be very much shortened; for, as greater number and smaller number may mean any two numbers, two letters, as a and b, may be used to represent them; and $2a$ may represent twice the greater, and $2b$ twice the smaller. Then these statements become:

1. $(a+b)+(a-b)=2a.$

2. $(a+b)-(a-b)=2b.$

In studying the general properties of numbers, letters may represent any numerical values consistent with the conditions of the problem.

10. It is also convenient to use letters to denote numbers which are *unknown*, and which are to be found from certain given relations to other known numbers.

Thus, the solution of the problem, "Find two numbers such that, when the greater is divided by the less, the quotient is 4, and the remainder 3; and when the sum of the two numbers is increased by 38, and the result divided by the greater of the two numbers, the quotient is 2 and the remainder 2," is much simplified by the use of letters to represent the unknown numbers.

11. The science which employs letters in reasoning about numbers, either to discover their *general properties*, or to find the value of an *unknown number* from its relations to known numbers, is called **Algebra.**

Algebraic Numbers.

12. There are quantities which stand to each other in such opposite relations that, when we combine them, they cancel each other entirely or in part. Thus, six dollars *gain* and six dollars *loss* just cancel each other; but ten dollars *gain* and six dollars *loss* cancel each other only in part. For the six dollars *loss* will cancel six dollars of the *gain* and will leave four dollars.

An opposition of this kind exists in *assets* and *debts*, in *income* and *outlay*, in motion *forwards* and *backwards*, in motion *to the right* and *to the left*, in time *before* and *after* a fixed date, in the degrees *above* and *below* zero on a thermometer.

From this relation of quantities a question often arises which is not considered in Arithmetic; namely, the subtracting of a greater number from a smaller. This cannot be done in Arithmetic, for the real nature of subtraction consists in *counting backwards*, along the natural series of numbers. If we wish to subtract four from six, we start at six in the natural series, count four units backwards, and

arrive at two, the difference sought. If we subtract six from six, we start at six in the natural series, count six units backwards, and arrive at zero. If we try to subtract nine from six, we cannot do it, because, when we have counted backwards as far as zero, *the natural series of numbers comes to an end.*

13. In order to subtract a greater number from a smaller it is necessary to *assume* a new series of numbers, beginning at zero and extending to the left of zero. The series to the left of zero must ascend from zero by the repetitions of the unit, precisely like the natural series to the right of zero; and the *opposition* between the right-hand series and the left-hand series must be clearly marked. This opposition is indicated by calling every number in the right-hand series a *positive* number, and prefixing to it, when written, the sign $+$; and by calling every number in the left-hand series a *negative* number, and prefixing to it the sign $-$. The two series of numbers will be written thus:

$$\cdots\cdots -4,\ -3,\ -2,\ -1,\quad 0,\ +1,\ +2,\ +3,\ +4, \cdots\cdots$$

If, now, we wish to subtract 9 from 6, we begin at 6 in the positive series, count nine units in the *negative direction* (to the left), and arrive at -3 in the negative series. That is, $6-9=-3$.

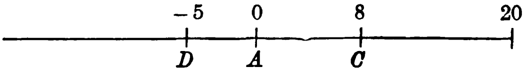
The result obtained by subtracting a greater number from a less, when both are positive, is *always a negative number.*

If a and b represent any two numbers of the positive series, the expression $a-b$ will denote a positive number when a is greater than b; will be equal to zero when a is equal to b; will denote a negative number when a is less than b.

If we wish to add 9 to -6, we begin at -6, in the

negative series, count nine units in the *positive direction* (to the right), and arrive at $+3$, in the positive series.

We may illustrate the use of positive and negative numbers as follows:



Suppose a person starting at A walks 20 feet to the right of A, and then returns 12 feet, where will he be? Answer: at C, a point 8 feet to the right of A. That is, 20 feet $-$ 12 feet $=$ 8 feet; or, $20-12=8$.

Again, suppose he walks from A to the right 20 feet, and then returns 20 feet, where will he be? Answer: at A, the point from which he started. That is, $20-20=0$.

Again, suppose he walks from A to the right 20 feet, and then returns 25 feet, where will he now be? Answer: at D, a point 5 feet to the left of A. That is, if we consider distance measured in feet to the left of A as forming a negative series of numbers, beginning at A, $20-25=-5$. Hence, the phrase, 5 feet to the left of A, is now expressed by the negative number -5.

14. Numbers provided with the sign $+$ or $-$ are called **algebraic numbers.** They are unknown in Arithmetic, but play a very important part in Algebra. In contradistinction, numbers not affected by the signs $+$ or $-$ are termed **absolute numbers.**

15. Every algebraic number, as $+4$ or -4, consists of a sign $+$ or $-$ and the absolute value of the number; in this case 4. The sign shows whether the number belongs to the positive or negative series of numbers; the absolute value shows what place the number has in the positive or negative series.

16. When no sign stands before a number, the sign $+$ is always understood; thus, 4 means the same as $+4$, a means the same as $+a$. But *the sign $-$ is never omitted.*

17. Two numbers which have one the sign $+$ and the other the sign $-$, are said to have **unlike signs.**

18. Two numbers which have the same absolute values, but unlike signs, always cancel each other when combined; thus $+4-4=0$, $+a-a=0$.

19. The use of the signs $+$ and $-$, to indicate addition and subtraction, must be carefully distinguished from their use to indicate in which series, the positive or the negative, a given number belongs. In the first sense, they are signs of *operations*, and are common to both Arithmetic and Algebra. In the second sense, they are signs of *opposition*, and are employed in Algebra alone.

FACTORS AND POWERS.

20. When a number consists of the product of two or more numbers, each of these numbers is called a **factor** of the product.

When these numbers are denoted by letters, the sign $\times$ is omitted; thus, instead of $a \times b$, we write ab; instead of $a \times b \times c$, we write abc.

The expression abc must not be confounded with $a+b+c$; the first is a product, the second is a sum. If $a=2$, $b=3$, $c=4$, then

$$abc = 2 \times 3 \times 4 = 24;$$

$$a+b+c = 2+3+4 = 9.$$

21. Factors expressed by letters are called **literal** factors; factors expressed by figures are called **numerical** factors.

22. A known factor of a product which is prefixed to another factor, to show how many times that factor is taken, is called a **coefficient.** Thus, in $7c$, 7 is the coefficient of c; in $7ax$, 7 is the coefficient of ax, or, if a be known, $7a$ is the coefficient of x. When no numerical coefficient occurs in a product, 1 is always understood. Thus, ax means the same as $1ax$.

23. A product consisting of two or more **equal** factors is called a **power** of that factor.

24. The **index** or **exponent of a power** is a small figure or letter placed at the right of a number, to show how many times the number is taken as a factor. Thus, a^1, or simply a, denotes that a is taken once as a factor; a^2 denotes that a is taken twice as a factor; a^3 denotes that a is taken three times as a factor; and a^n denotes that a is taken n times as a factor. These are read: the first power of a; the second power of a; the third power of a; the nth power of a.

a^3 is written instead of aaa.

a^n is written instead of aaa, etc., repeated n times.

The meaning of coefficient and exponent must be carefully distinguished. Thus,

$$4a = a + a + a + a;$$

$$a^4 = a \times a \times a \times a.$$

If $a = 3$,

$$4a = 3 + 3 + 3 + 3 = 12.$$

$$a^4 = 3 \times 3 \times 3 \times 3 = 81.$$

25. The second power of a number is generally called the *square* of that number; thus, a^2 is called the *square* of a, because if a denote the number of units of length in the side of a square, a^2 denotes the number of units of surface in the square.

The third power of a number is generally called the *cube* of that number; thus, a^3 is called the *cube* of a, because if a denote the number of units of length in the edge of a cube, a^3 denotes the number of units of volume in the cube.

ALGEBRAIC SYMBOLS.

26. Known numbers in Algebra are denoted by figures and by the first letters of some alphabet; as, a, b, c, etc.; a', b', c', read *a prime*, *b prime*, *c prime*, etc.; a_1, b_1, c_1, read *a one*, *b one*, *c one*.

Unknown numbers are generally denoted by the last letters of some alphabet; as, x, y, z, x', y', z', etc.

27. The **symbols of operations** are the same in Algebra as in Arithmetic. One point of difference, however, must be carefully observed. When a symbol of operation is omitted in the notation of Arithmetic, it is always the *symbol of addition;* but when a symbol of operation is omitted in the notation of Algebra, it is always the *symbol of multiplication.* Thus, 456 means $400 + 50 + 6$, but $4ab$ means $4 \times a \times b$; $4\frac{5}{6}$ means $4 + \frac{5}{6}$, but $4\dfrac{a}{b}$ means $4 \times \dfrac{a}{b}$.

28. The **symbols of relation** are $=$, $>$, $<$, which stand for the words, "is equal to," "is greater than," and "is less than," respectively.

29. The **symbols of aggregation** are the bar, $|$; the vinculum, $\overline{\quad}$; the parenthesis, ( ); the bracket, [ ]; and the brace, { }. Thus, each of the expressions, $\left.\begin{array}{r} x \\ +y \end{array}\right|$, $\overline{x+y}$, $(x+y)$, $[x+y]$, $\{x+y\}$, signifies that $x+y$ is to be treated as a single number.

30. The **symbols of continuation** are dots, ..., or dashes, ------, and are read, "and so on."

31. The **symbol of deduction** is $\therefore$, and is read, "hence," or "therefore."

Algebraic Expressions.

32. An **algebraic expression** is any number written in algebraic symbols. Thus, $8c$ is the algebraic expression for 8 times the number denoted by c.

$7a^2 - 3ab$ is the algebraic expression for 7 times the square of the number denoted by a, diminished by 3 times the product of the numbers denoted by a and b.

33. A **term** is an algebraic expression the parts of which are not separated by the sign of addition or subtraction. Thus, $3ab$, $5xy$, $3ab \div 5xy$ are terms.

34. A **monomial** or **simple expression** is an expression which contains only one term.

35. A **polynomial** or **compound expression** is an expression which contains two or more terms. A **binomial** is a polynomial of two terms. A **trinomial** is a polynomial of three terms.

36. **Like terms** are terms which have the same letters, and the corresponding letters affected by the same exponents. Thus, $7a^2cx^3$ and $-5a^2cx^3$ are like terms; but $7a^2cx^3$ and $-5ac^2x^3$ are unlike terms.

37. The **dimensions** of a term are its literal factors.

38. The **degree** of a term is equal to the number of its dimensions, and is found by taking the sum of the exponents of its literal factors. Thus, $3xy$ is of the *second* degree, and $5x^2yz^3$ is of the *sixth* degree.

39. A polynomial is said to be **homogeneous** when all its terms are of the same degree. Thus, $7x^3 - 5x^2y + xyz$ is homogeneous, for each term is of the third degree.

40. A polynomial is said to be **arranged** according to the powers of some letter when the exponents of that letter either descend or ascend in order of magnitude. Thus, $3ax^3 - 4bx^2 - 6ax + 8b$ is arranged according to the descending powers of x, and $8b - 6ax - 4bx^2 + 3ax^3$ is arranged according to the ascending powers of x.

41. The **numerical value** of an algebraic expression is the number obtained by giving a particular value to each letter, and then performing the operations indicated.

42. Two numbers are **reciprocals** of each other when their product is equal to unity. Thus, a and $\frac{1}{a}$ are reciprocals.

Axioms.

43. 1. Things which are equal to the same thing are equal to each other.

2. If equal numbers be added to equal numbers, the sums will be equal.

3. If equal numbers be subtracted from equal numbers, the remainders will be equal.

4. If equal numbers be multiplied into equal numbers, the products will be equal.

5. If equal numbers be divided by equal numbers, the quotients will be equal.

6. If the same number be both added to and subtracted from another, the value of the latter will not be altered.

7. If a number be both multiplied and divided by another, the value of the former will not be altered.

Exercise I.

If $a=1,\ b=2,\ c=3,\ d=4,\ e=5,\ f=0$, find the numerical values of the following expressions:

1. $9a+2b+3c-2f$.

2. $4e-3a-3b+5c$.

3. $8abc-bcd+9cde-def$.

4. $\dfrac{4ac}{b}+\dfrac{8bc}{d}-\dfrac{5cd}{e}$.

5. $7e+bcd-\dfrac{3bde}{2ac}$.

6. $abc^2+bcd^2-dea^2+f^2$.

7. $e^4+6e^2b^2+b^4-4e^3b-4eb^3$.

8. $\dfrac{8a^2+3b^2}{a^2b^2}+\dfrac{4c^2+6b^2}{c^2-b^2}-\dfrac{c^2+d^2}{e^2}$.

9. $\dfrac{d^c}{b^c}$.

10. $\dfrac{e^c+b^a}{c^b-b^c}$.

11. $\dfrac{b^c+d^c}{b^2+d^2-bd}$.

12. $\dfrac{e^c-d^c}{e^2+ed+d^2}$.

In simplifying compound expressions, each term must be reduced to its simplest form before the operations of addition and subtraction are performed.

Simplify the following expressions:

13. $100+80\div 4$.

14. $75-25\times 2$.

15. $25+5\times 4-10\div 5$.

16. $24-5\times 4\div 10+3$

17. $(24-5)\times(4\div 10+3)$.

Find the numerical value of the following expressions, in which $a=2,\ b=10,\ x=3,\ y=5$:

18. $xy+4a\times 2$.

19. $xy-15b\div 5$.

20. $3x+7y\div 7+a\times y$.

21. $6b-8y\div 2y\times b-2b$.

22. $(6b - 8y) \div 2y \times b + 2b$.

23. $(6b - 8y) \div (2y \times b) + 2b$.

24. $6b - (8y \div 2y) \times b - 2b$.

25. $6b \div (b - y) - 3x + bxy \div 10a$.

Algebraic Notation.

26. Express the sum of a and b.

27. Express the double of x.

28. By how much is a greater than 5?

29. If x be a whole number, what is the next number above it?

30. Write five numbers in order of magnitude, so that x shall be the middle number.

31. What is the sum of $x + x + x + \cdots$ written a times?

32. If the product be xy and the multiplier x, what is the multiplicand?

33. A man who has a dollars spends b dollars; how many dollars has he left?

34. A regiment of men can be drawn up in a ranks of b men each, and there are c men over; of how many men does the regiment consist?

35. Write, the sum of x and y divided by c is equal to the product of a, b, and m, diminished by six times c, and increased by the quotient of a divided by the sum of x and y.

36. Write, six times the square of n, divided by m minus a, increased by five b into the expression c plus d minus a.

37. Write, four times the fourth power of a, diminished by five times the square of a into the square of b, and increased by three times the fourth power of b.

EXERCISE II.

That the beginner may see how Algebra is employed in the solution of problems, the following simple exercises are introduced:

1. John and James together have \$6. James has twice as much as John. How much has each?

Let x denote the *number* of dollars John has.

Then $2x =$ *number* of dollars James has,
and $x + 2x =$ number of dollars both have.

But $6 =$ number of dollars both have;

$$\therefore x + 2x = 6,$$

or $$3x = 6.$$

and $$x = 2.$$

Therefore, John has \$2, and James has \$4.

2. A stick of timber 40 feet long is sawed in two, so that one part is two-thirds as long as the other. Required the length of each part.

Let $3x$ denote the *number* of feet in the longer part.

Then $2x =$ *number* of feet in the shorter part,
and $3x + 2x =$ number of feet in both together.

But $40 =$ number of feet in both together;

$$\therefore 3x + 2x = 40,$$

or $$5x = 40,$$

and $$x = 8.$$

Therefore, the longer part, or $3x$, is 24 feet long; and the shorter, or $2x$, is 16 feet.

NOTE. The *unit* of the quantity sought is always given, and only the **number** of such units is required. Therefore, x must never be put for *money*, *length*, *time*, *weight*, etc., but always for the required **number** *of specified units* of money, length, time, weight, etc.

The beginner should give particular attention to this caution.

3. The greater of two numbers is six times the smaller, and their sum is 35. Required the numbers.

4. Thomas had 75 cents. After spending a part of his money, he found he had twice as much left as he had spent. How much had he spent?

5. A tree 75 feet high was broken, so that the part broken off was four times the length of the part left standing. Required the length of each part.

6. Four times the smaller of two numbers is three times the greater, and their sum is 63. Required the numbers.

7. A farmer sold a sheep, a cow, and a horse, for $216. He sold the cow for seven times as much as the sheep, and the horse for four times as much as the cow. How much did he get for each?

8. George bought some apples, pears, and oranges, for 91 cents. He paid twice as much for the pears as for the apples, and twice as much for the oranges as for the pears. How much money did he spend for each?

9. A man bought a horse, wagon, and harness, for $350. He paid for the horse four times as much as for the harness, and for the wagon one-half as much as for the horse. What did he pay for each?

10. Distribute $3 among Thomas, Richard, and Henry, so that Thomas and Richard shall each have twice as much as Henry.

11. Three men, A, B, and C, pay $1000 taxes. B pays 4 times as much as A, and C an amount equal to the sum of what the other two pay. How much does each pay?

CHAPTER II.

ADDITION AND SUBTRACTION.

44. AN algebraic number which is to be added or subtracted is often inclosed in a parenthesis, in order that the signs $+$ and $-$ which are used to distinguish positive and negative numbers may not be confounded with the $+$ and $-$ signs that denote the operations of addition and subtraction. Thus, $+4+(-3)$ expresses the sum, and $+4-(-3)$ expresses the difference, of the numbers $+4$ and -3.

45. In order to add two algebraic numbers, we begin at *the place in the series which the first number occupies*, and count, *in the direction indicated by the sign of the second number*, as many units as are equal to the absolute value of the second number. Thus, the sum of $+4+(+3)$ is found by counting from $+4$ three units in the *positive* direction, and is, therefore, $+7$; the sum of $+4+(-3)$ is found by counting from $+4$ three units in the *negative* direction, and is, therefore, $+1$.

In like manner, the sum of $-4+(+3)$ is -1, and the sum of $-4+(-3)$ is -7. That is,

(1) $+4+(+3)=7$; (3) $-4+(+3)=-1$;

(2) $+4+(-3)=1$; (4) $-4+(-3)=-7$.

I. Therefore, to add two numbers with **like** signs, *find the* **sum** *of their absolute values, and prefix the common sign to the sum.*

II. To add two numbers with **unlike** signs, *find the* **difference** *of their absolute values, and prefix the sign of the greater number to the difference.*

EXERCISE III.

1. $+16+(-11)=$
3. $+68+(-79)=$
2. $-15+(-25)=$
4. $-7+(+4)=$
5. $+33+(+18)=$
6. $+378+(+709)+(-592)=$
7. A man has \$5242 and owes \$2758. How much is he worth?
8. The First Punic War began B.C. 264, and lasted 23 years. When did it end?
9. Augustus Cæsar was born B.C. 63, and lived 77 years. When did he die?
10. A man goes 65 steps forwards, then 37 steps backwards, then again 48 steps forwards. How many steps did he take in all? How many steps is he from where he started?

ADDITION OF MONOMIALS.

46. If a and b denote the absolute values of any two numbers, 1, 2, 3, 4 (§ 45) become:

(1) $+a+(+b)=a+b$; (3) $-a+(+b)=-a+b$;

(2) $+a+(-b)=a-b$; (4) $-a+(-b)=-a-b$.

Therefore, to add two terms, *write them one after the other with unchanged signs.*

It should be noticed that the order of the terms is immaterial. Thus, $+a-b=-b+a$. If $a=8$ and $b=12$, the result in either case is -4.

47.
$$3a+5a+2a+6a+a=17a.$$
$$-2c-3c-c-4c-8c=-18c.$$

Therefore, to add several **like terms** which have the **same**

sign, *add the coefficients, prefix the common sign, and annex the common symbols.*

48. $7a - 6a + 11a + a - 5a - 2a = 19a - 13a = 6a.$

$-3a - 15a - 7a + 14a - 2a = 14a - 27a = -13a.$

Therefore, to add several **like terms** which have **not all the same sign,** *find the difference between the sum of the positive coefficients and the sum of the negative coefficients, prefix the sign of the greater sum, and annex the common symbols.*

49. $5a - 2b + 3a = 8a - 2b.$

$-3ax + 8y + 9ax - 4c = 6ax + 8y - 4c.$

Therefore, to add terms which are **not all like terms,** *combine the like terms, and write down the other terms, each preceded by its proper sign.*

EXERCISE IV.

1. $5ab + (-5ab) =$
2. $8mx + (-2mx) =$
3. $-13mng + (-7mng) =$
4. $-5x^2 + (+8x^2) =$
5. $25my^2 + (-18my^2) =$
6. $7ab + (-5ab) =$
7. $120my + (-95my) =$
8. $-33ab^2 + (11ab^2) =$
9. $-75xy + (+20xy) =$
10. $+15a^2x^2 + (-a^2x^2) =$
11. $-b^2m^3 + (+7b^2m^3) =$
12. $5a + (-3b) + (+4a) + (-7b) =$
13. $4a^2c + (-10xyz) + (+6a^2c) + (-9xyz) + (-11a^2c) + (+20xyz) =$
14. $3x^2y + (-4ab) + (-2mn) + (+5x^2y) + (-x^2y) + (-4x^2y) =$

ADDITION OF POLYNOMIALS.

50. Two or more polynomials are added by adding their separate terms.

It is convenient to arrange the terms in columns, so that like terms shall stand in the same column. Thus,

$$
(1)\quad \begin{array}{rrrr}
2a^3 & -\,3a^2b & +\,4ab^2 & +\;\;b^3 \\
a^3 & +\,4a^2b & -\,7ab^2 & -\,2b^3 \\
-\,3a^3 & +\;\;a^2b & -\,3ab^2 & -\,4b^3 \\
2a^3 & +\,2a^2b & +\,6ab^2 & -\,3b^3 \\
\hline
2a^3 & +\,4a^2b & & -\,8b^3
\end{array}
$$

$$
(2)\quad \begin{array}{rrrr}
-\,2x^2y & & +\,6y^3 & -\,1 \\
-\,4x^2y & +\,2xy^2 & & +\,5 \\
6x^2y & & & +\,2 \\
x^2y & & -\;\;y^3 & \\
-\,2x^2y & & & -\,5 \\
\hline
-\;\;x^2y & +\,2xy^2 & +\,5y^3 & +\,1
\end{array}
$$

EXERCISE V.

Add:

1. $5a+3b+c,\quad 3a+3b+3c,\quad a+3b+5c.$

2. $7a-4b+c,\quad 6a+3b-5c,\quad -12a+4c.$

3. $a+b-c,\quad b+c-a,\quad c+a-b,\quad a+b-c.$

4. $a+2b+3c,\quad 2a-b-2c,\quad b-a-c,\quad c-a-b.$

5. $a-2b+3c-4d,\quad 3b-4c+5d-2a,$
$5c-6d+3a-4b,\quad 7d-4a+5b-4c.$

6. $x^3-4x^2+5x-5,\quad 2x^3-7x^2-7x^2-14x+5,$
$-x^3+9x^2+x+8.$

7. $x^4-2x^3+3x^2,\quad x^3+x^2+x,\quad 4x^4+5x^3,$
$2x^2+3x-4,\quad -3x^2-2x-5.$

8. $a^3+3ab^2-3a^2b-b^3,\quad 2a^3+5a^2b-6ab^2-7b^3,$
$a^3-ab^2+2b^3.$

9. $2ab-3ax^2+2a^2x,\quad 12ab-6a^2x+10ax^2,$
$ax^2-8ab-5a^2x.$

10. $c^4-3c^3+2c^2-4c+7,\quad 2c^4+3c^3+2c^2+5c+6,$
$-4c^4-4c^2-5.$

11. $3x^2-xy+xz-3y^2+4yz-z^2,\quad -5x^2-xy-xz+5yz,$
$6x^2-6y-6z,\quad 4yz-5yz+3z^2,$
$-4x^2+y^2+3yz+3z^2.$

12. $m^5-3m^4n-6m^3n^2,\quad +m^3n^2+m^2n^3-5m^4n,$
$7m^3n^2+4m^2n^3-3mn^4,\quad -2m^2n^3-3mn^4+4n^5,$
$2mn^4+2n^5+3m^5,\quad -n^5+2m^5+7m^4n.$

Subtraction.

51. In order to find the difference between two algebraic numbers, we begin *at the place in the series which the minuend occupies*, and *count in the direction opposite to that indicated by the sign of the subtrahend* as many units as are equal to the absolute value of the subtrahend.

Thus, the difference between $+4$ and $+3$ is found by counting from $+4$ three units in the *negative* direction, and is, therefore, $+1$; the difference between $+4$ and -3 is found by counting from $+4$ three units in the *positive* direction, and is, therefore, $+7$.

In like manner, the difference between -4 and $+3$ is -7; the difference between -4 and -3 is -1.

Compare these results with results obtained in addition:

(1) $+4-(+3)=1$	$+4+(-3)=1.$
(2) $+4-(-3)=7$	$+4+(+3)=7.$
(3) $-4-(+3)=-7$	$-4+(-3)=-7.$
(4) $-4-(-3)=-1$	$-4+(+3)=-1.$

Or, (1) $+4-(+3)=+4+(-3).$
(2) $+4-(-3)=+4+(+3).$
(3) $-4-(+3)=-4+(-3).$
(4) $-4-(-3)=-4+(+3).$

52. From (1) and (3), it is evident that *subtracting a positive number is equivalent to adding an equal negative number.*

From (2) and (4), it is evident that *subtracting a negative number is equivalent to adding an equal positive number.*

To subtract, therefore, one algebraic number from another, *change the sign of the subtrahend, and then add the subtrahend to the minuend.*

Exercise VI.

1. $+25-(+16)=$ 3. $-31-(+58)=$

2. $-50-(-25)=$ 4. $+107-(-93)=$

5. Rome was ruled by emperors from B.C. 30 to its fall, A.D. 476. How long did the empire last?

6. The continent of Europe lies between 36° and 71° north latitude, and between 12° west and 63° east longitude (from Paris). How many degrees does it extend in latitude, and how many in longitude?

Subtraction of Monomials.

If a and b denote the absolute values of any two numbers, 1, 2, 3, and 4 (§ 51) become:

(1) $+a-(+b)=a-b.$ (3) $-a-(+b)=-a-b.$

(2) $+a-(-b)=a+b.$ (4) $-a-(-b)=-a+b.$

To subtract, therefore, one term from another, *change the sign of the term to be subtracted, and write the terms one after the other.*

Exercise VII.

1. $5x-(-4x)=$
2. $-3ab-(+5ab)=$
3. $3ab^2-(+10ab^2)=$
4. $15m^2x^2-(-7m^2x^2)=$
5. $-7ay-(-3ay)=$
6. $17ax^3-(-24ax^3)=$
7. $5a^2x-(-3a^2x)=$
8. $-4xy-(-5xy)=$
9. $8ax-(-3ay)=$
10. $2ab^2y-(+aby)=$
11. $9x^2+(5x^2)-(+8x^2)=$
12. $5x^2y-(-18x^2y)+(-10x^2y)=$
13. $17ax^3-(-ax^3)-(+24ax^3)=$
14. $-3ab+(2mx)-(-4mx)=$
15. $3a-(+2b)-(-4c)=$

Subtraction of Polynomials.

53. When one polynomial is to be subtracted from another, place its terms under the like terms of the other, change the signs of the subtrahend, and add.

$$\begin{array}{ll} \text{From} & 4x^3-3x^2y-\ \ xy^2+2y^3 \\ \text{take} & \underline{2x^3-\ \ x^2y+5xy^2-3y^3} \end{array}$$

Change the signs of the subtrahend and add:

$$\begin{array}{r} 4x^3-3x^2y-\ \ xy^2+2y^3 \\ -2x^3+\ \ x^2y-5xy^2+3y^3 \\ \hline 2x^3-2x^2y-6xy^2+5y^3 \end{array}$$

$$\begin{array}{lr} \text{From} & a^3x^2+2a^2x^3-4ax^4 \\ \text{take} & a^5+4a^3x^2-3a^2x^3-4ax^4 \\ \hline & -a^5-3a^3x^2+5a^2x^3\qquad\qquad \end{array}$$

In the last example we have conceived the signs to be changed without actually changing them. The beginner should do the examples by both methods until he has acquired sufficient practice, when he should use the second method only.

EXERCISE VIII.

1. From $6a-2b-c$ take $2a-2b-3c$.

2. From $3a-2b+3c$ take $2a-7b-c-b$.

3. From $7x^2-8x-1$ take $5x^2-6x+3$.

4. From $4x^4-3x^3-2x^2-7x+9$ take $x^4-2x^3-2x^2+7x-9$.

5. From $2x^2-2ax+3a^2$ take x^2-ax+a^2.

6. From $x^2-3xy-y^2+yz-2z^2$ take $x^2+2xy+5xz-3y^2-2z^2$.

7. From $a^3-3a^2b+3ab^2-b^3$ take $-a^3+3a^2b-3ab^2+b^3$.

8. From $x^2-5xy+xz-y^2+7yz+2z^2$ take $x^2-xy-xz+2yz+3z^2$.

9. From $2ax^2+3abx-4b^2x+12b^3$ take $ax^2-4abx+bx^2-5b^2x-x^3$.

10. From $6x^3-7x^2y+4xy^2-2y^3-5x^2+xy-4y^2+2$ take $8x^3-7x^2y+xy^2-y^3+9x^2-xy+6y^2-4$.

11. From a^4-b^4 take $4a^3b-6a^2b^2+4ab^3$, and from the result take $2a^4-4a^3b+6a^2b^2+4ab^3-2b^4$.

12. From $x^3y^2-3x^2y^3+4xy^4-y^5$ take $-x^5+2x^4y-4xy^4-4y^5$. Add the same two expressions, and subtract the former result from the latter.

13. From $a^2b^2-a^2bc-8ab^2c-a^2c^2+abc^2-6b^2c^2$ take $2a^2bc-5ab^2c+2abc^2-5b^2c^2$.

14. From $12a+3b-5c-2d$ take $10a-b+4c-3d$, and show that the result is numerically correct when $a=6$, $b=4$, $c=1$, $d=5$.

15. What number must be added to a to make b; and what number must be taken from $2a^3-6a^2b+6ab^2-2b^3$ to leave $a^3-7a^2b-3b^3$?

16. From $2x^2-y^2-2xy+z^2$ take $x^2-y^2+2xy-z^2$.

17. From $12ac+8cd-9$ take $-7ac-9cd+8$.

18. From $-6a^2+2ab-3c^2$ take $4a^2+6ab-4c^2$.

19. From $9xy-4x-3y+7$ take $8xy-2x+3y+6$.

20. From $-a^2bc-ab^2c+abc^2-abc$
take $a^2bc+ab^2c-abc^2+abc$.

21. From $7x^2-2x+4$ take $2x^2+3x-1$.

22. From $3x^2+2xy-y^2$ take $-x^2-3xy+3y^2$, and from the remainder take $3x^2+4xy-5y^2$.

23. From ax^2-by^2 take cx^2-dy^2.

24. From $ax+bx+by+cy$ take $ax-bx-by+cy$.

25. From $5x^2+4x-4y+3y^2$ take $5x^2-3x+3y+y^2$.

26. From $a^2b^2+12abc-9ax^2$ take $4ab^2-6acx+3a^2x$.

27. From $a^2-2ab+c^2-3b^2$ take $2a^2-2ab+3b^2$.

28. From the sum of the first four of the following expressions, $a^2+b^2+c^2+d^2$, $d^2+b^2+c^2$, $a^2-c^2+b^2-d^2$, $a^2-b^2+c^2+d^2$, $b^2+c^2+d^2-a^2$, take the sum of the last four.

29. From $2x^2-2y^2-z^2$ take $3y^2+2x^2-z^2$, and from the remainder take $3z^2-2y^2-x^2$.

30. From $a^3-2a^2c+3ac^2$ take the sum of $a^2c-2a^3+2ac^2$ and $a^3-ac^2-a^2c$.

Parentheses.

54. From (§ 52), it appears that

$$(1)\quad a+(+b)=a+b.$$

$$(2)\quad a+(-b)=a-b.$$

$$(3)\quad a-(+b)=a-b.$$

$$(4)\quad a-(-b)=a+b.$$

The same laws respecting the removal of parentheses hold true whether one or more terms are inclosed. Hence, when an expression within a parenthesis is preceded by a **plus sign,** the parenthesis may be removed.

When an expression within a parenthesis is preceded by a **minus sign,** the parenthesis may be removed *if the sign of every term within the parenthesis be changed.* Thus:

$$(1)\quad a+(b-c)=a+b-c.$$

$$(2)\quad a-(b-c)=a-b+c.$$

55. Expressions may occur with more than one parenthesis. These parentheses may be removed in succession, by removing *first, the innermost parenthesis;* next, the innermost of all that remain, and so on. Thus:

$$\begin{aligned}(1)\quad & a-\{b-(c-d)\}\\ &=a-\{b-c+d\},\\ &=a-b+c-d.\end{aligned}$$

$$\begin{aligned}(2)\quad & a-[b-\{c+(d-\overline{e-f})\}]\\ &=a-[b-\{c+(d-e+f)\}],\\ &=a-[b-\{c+d-e+f\}],\\ &=a-[b-c-d+e-f],\\ &=a-b+c+d-e+f.\end{aligned}$$

Exercise IX.

Simplify the following expressions by removing the parentheses and combining like terms.

1. $(a+b)+(b+c)-(a+c)$.
2. $(2a-b-c)-(a-2b+c)$.
3. $(2x-y)-(2y-z)-(2z-x)$.
4. $(a-x-y)-(b-x+y)+(c+2y)$.
5. $(2x-y+3z)+(-x-y-4z)-(3x-2y-z)$.
6. $(3a-b+7c)-(2a+3b)-(5b-4c)+(3c-a)$.
7. $1-(1-a)+(1-a+a^2)-(1-a+a^2-a^3)$.
8. $a-\{2b-(3c+2b)-a\}$.
9. $2a-\{b-(a-2b)\}$.
10. $3a-\{b+(2a-b)-(a-b)\}$.
11. $7a-[3a-\{4a-(5a-2a)\}]$.
12. $2x+(y-3z)-\{(3x-2y)+z\}+5x-(4y-3z)$.
13. $\{(3a-2b)+(4c-a)\}-\{a-(2b-3a)-c\}$
    $+\{a-(b-5c-a)\}$.
14. $a-[2a+(3a-4a)]-5a-\{6a-[(7a+8a)-9a]\}$.
15. $2a-(3b+2c)-[5b-(6c-6b)+5c$
    $-\{2a-(c+2b)\}]$.
16. $a-[2b+\{3c-3a-(a+b)\}+\{2a-(b+c)\}]$.
17. $16-x-[7x-\{8x-(9x-\overline{3x-6x})\}]$.
18. $2a-[3b+(2b-c)-4c+\{2a-(3b-\overline{c-2b})\}]$.
19. $a-[2b+\{3c-3a-(a+b)\}+2a-(b+3c)]$.
20. $a-[5b-\{a-(3c-3b)+2c-(a-2b-c)\}]$.

56. The rules for introducing parentheses follow directly from the rules for removing them:

1. Any number of terms of an expression may be put within a parenthesis, and the sign **plus** placed before the whole.

2. Any number of terms of an expression may be put within a parenthesis, and the sign **minus** placed before the whole; *provided the sign of every term within the parenthesis be changed.*

It is usual to prefix to the parenthesis the sign of the first term that is to be inclosed within it.

Exercise X.

Express in binomials, and also in trinomials:

1. $2a - 3b - 4c + d + 3e - 2f.$
2. $a - 2x + 4y - 3z - 2b + c.$
3. $a^5 + 3a^4 - 2a^3 - 4a^2 + a - 1.$
4. $-3a - 2b + 2c - 5d - e - 2f.$
5. $ax - by - cz - bx + cy + az.$
6. $2x^5 - 3x^4y + 4x^3y^2 - 5x^2y^3 + xy^4 - 2y^5.$
7. Express each of the above in trinomials, having the last two terms inclosed by *inner* parentheses.

Collect in parentheses the coefficients of x, y, z in

8. $2ax - 6ay + 4bz - 4bx - 2cx - 3cy.$
9. $ax - bx + 2ay + 3y + 4az - 3bz - 2z.$
10. $ax - 2by + 5cz - 4bx - 3cy + az - 2cx - ay + 4bz.$
11. $12ax + 12ay + 4by - 12bz - 15cx + 6cy + 3cz.$
12. $2ax - 3by - 7cz - 2bx + 2cx + 8cz - 2cx - cy - cz.$

CHAPTER III.

MULTIPLICATION OF ALGEBRAIC NUMBERS.

57. THE operation of finding the sum of 3 numbers, each equal to 5, is symbolized by the expression, $3 \times 5 = 15$, read, "three times five is equal to fifteen"; or, by the expression $5 \times 3 = 15$, read, "five multiplied by three is equal to fifteen."

58. With reference to this operation, this sum is called the **product**; one of the equal numbers is called the **multiplicand**; and the number which shows how many times the multiplicand is to be taken is called the **multiplier.**

59. The multiplier means so many **times.** The multiplicand can be a *positive* or a *negative number;* but the multiplier can only mean that the multiplicand is taken *so many times to be added*, or *so many times to be subtracted.*

60. If we have to multiply 867 by 98, we may put the multiplier in the form $100 - 2$. The 100 will mean that the multiplicand is taken 100 times *to be added;* the -2 will mean that the multiplicand is taken twice *to be subtracted.*

In general, a multiplier with $+$ before it, expressed or understood, means that the multiplicand is taken so many times *to be added;* and a multiplier with $-$ before it means that the multiplicand is taken so many times *to be subtracted.* Thus,

(1) $+3 \times (+5) = (+5) + (+5) + (+5)$, or $(+15)$.

(2) $+3 \times (-5) = (-5) + (-5) + (-5)$, or (-15).

(3) $-3 \times (+5) = -(+5) - (+5) - (+5)$, or (-15).

(4) $-3 \times (-5) = -(-5) - (-5) - (-5)$, or $(+15)$.

From these four cases it follows, that, in finding the product of two numbers,

61. *Like* signs produce **plus**; *unlike* signs, **minus**.

Exercise XI.

1. $-17 \times 8 =$
2. $-12.8 \times 25 =$
3. $3.29 \times 5.49 =$
4. $-18 \times -5 =$
5. $43 \times -6 =$
6. $457 \times 100 =$
7. $(-358 - 417) \times -79 =$
8. $(7.512 - \{-2.894\}) \times (-6.037 + \{13.963\}) =$

62. The product of more than two factors, each preceded by $-$, will be positive or negative, according as the number of such factors is even or odd. Thus,

$$-2 \times -3 \times -4 = +6 \times -4 = -24.$$

$$-2 \times -3 \times -4 \times -5 = -24 \times -5 = +120.$$

9. $13 \times 8 \times -7 =$
10. $-38 \times 9 \times -6 =$
11. $-20.9 \times -1.1 \times 8 =$
12. $-78.3 \times -0.57 \times +1.38 \times -27.9 =$
13. $-2.906 \times -2.076 \times -1.49 \times 0.89 =$

MULTIPLICATION OF MONOMIALS.

63. The product of numerical factors is a new number in which no trace of the original factors is found. Thus, $4 \times 9 = 36$. But the product of literal factors can only be expressed by writing them one after the other. Thus, the product of a and b is expressed by ab; the product of ab and cd is expressed by $abcd$.

64. If we have to multiply $5a$ by $-4b$, the factors will give the same result in whatever order they are taken. Thus, $5a \times -4b = 5 \times -4 \times a \times b = -20 \times ab = -20ab$.

65. Hence, to find the product of monomials, *annex the literal factors to the product of the numerical factors.*

66. $a^2 \times a^3 = aa \times aaa = aaaaa = a^5$.

$$a^2 \times a^3 \times a^4 = aa \times aaa \times aaaa = aaaaaaaaa = a^9.$$

It is evident that the exponent of the product is equal to the sum of the exponents of the factors. Hence,

67. *The product of two or more powers of any number is that number with an exponent equal to the sum of the exponents of the factors.*

EXERCISE XII.

1. $+a \times +b = +ab$.
2. $+a \times -b = -ab$.
3. $-a \times +b = -ab$.
4. $-a \times -b = +ab$.
5. $7a \times 5b = 35ab$.
6. $-3p \times 8m = -24pm$.
7. $3a^2 \times -a^3 = -3a^5$.
8. $-3a \times 2a^5 = -6a^6$.
9. $6a \times -2a =$
10. $5mn \times 9m =$

11. $3ax \times -4by =$

12. $-8cm \times dn =$

13. $-7ab \times 2ac =$

14. $5m^2x \times 3mx^2 =$

15. $5a^m \times -2a^n =$

16. $3a^2x^2 \times 7a^3x^4 =$

17. $7a \times -4b \times -8c =$

18. $8ab^2 \times 3ac \times -4c^2 =$

19. $27ab \times -39mp \times 18ap =$

20. $6ab^2y^3 \times 2b^3y^3 \times -5a^2y =$

21. $7m^2x \times 3mx^2 \times -2mq =$

22. $-3pq^2 \times 6p^3q \times 8p^2q^3 =$

23. $2a^2m^3x^4 \times 3am^5x^2 \times 4a^3mx^2 =$

24. $6x^2yz^3 \times -9x^2y^2z^2 \times -3x^4yz =$

25. $3ax \times 2am \times -4mx \times b^2 =$

26. $7am^2 \times 3b^2n^2 \times -4ab \times a^2bn \times -2b^2n \times -mn^2 =$

Of Polynomials by Monomials.

68. If we have to multiply $a+b$ by n, that is, to take $(a+b)$ n times to be added, we have,

$$(a+b) \times n = (a+b) + (a+b) + (a+b) \ldots\ldots n \text{ times},$$
$$= a + a + a \ldots n \text{ times} + b + b + b \ldots\ldots n \text{ times},$$
$$= a \times n + b \times n,$$
$$= an + bn.$$

As it is immaterial in what order the factors are taken,

$$n \times (a+b) = an + bn.$$

In like manner,

$$(a+b+c) \times n = an + bn + cn,$$

or, $$n(a+b+c) = an + bn + cn.$$

Hence, to multiply a polynomial by a monomial,

69. ***Multiply each term of the polynomial by the monomial, and add the partial products.***

Exercise XIII.

1. $(6a - 5b) \times 3c = 18ac - 15bc.$
2. $(2 + 3a - 4a^2 - 5a^3)6a^2 = 12a^2 + 18a^3 - 24a^4 - 30a^5.$
3. $5a(3b + 4c - d) = 15ab + 20ac - 5ad.$
4. $-3ax(-by - 2cz + 5) = 3abxy + 6acxz - 15ax.$
5. $(4a^2 - 3b) \times 3ab =$
6. $(8a^3 - 9ab) \times 3a^2 =$
7. $(3x^2 - 4y^2 + 5z^2) \times 2x^2y =$
8. $(a^3x - 5a^2x^2 + ax^3 + 2x^4) \times ax^2y =$
9. $(-9a^5 + 3a^3b^2 - 4a^2b^3 - b^5) \times -3ab^4 =$
10. $(3x^3 - 2x^2y - 7xy^2 + y^3) \times -5x^2y =$
11. $(-4xy^2 + 5x^2y + 8x^3) \times -3x^2y =$
12. $(-3 + 2ab + a^2b^2) \times -a^4 =$
13. $(-z - 2xz^2 + 5x^2yz^2 - 6x^3y^2 + 3x^3y^2z) \times -3x^3yz =$

Of Polynomials by Polynomials.

70. If we have $a + b + c$ to be multiplied by $m + n + p$, we may represent the multiplicand $a + b + c$ by M. Then,

$$M(m + n + p) = M \times m + M \times n + M \times p.$$

If now we substitute for M its value,

$$\begin{aligned}(a + b + c)(m + n + p) = &(a + b + c) \times m \\ &+ (a + b + c) \times n \\ &+ (a + b + c) \times p;\end{aligned}$$

or, $$(a+b+c)(m+n+p) = am + bm + cm \\ + an + bn + cn \\ + ap + bp + cp.$$

That is, to find the product of two polynomials,

71. *Multiply the multiplicand by each term of the multiplier and add the partial products; or, multiply each term of one factor by each term of the other and add the partial products.*

Arrange the multiplicand and multiplier according to the descending powers of a common letter, and multiply:

22. $5x + 4x^2 + x^3 - 24$ by $x^2 + 11 - 4x$.

23. $x^3 + 11x - 4x^2 - 24$ by $x^2 + 5 + 4x$.

24. $x^4 + x^2 - 4x - 11 + 2x^3$ by $x^2 - 2x + 3$.

25. $-5x^4 - x^2 - x + x^5 + 13x^3$ by $x^2 - 2 - 2x$.

26. $3x + x^3 - 2x^2 - 4$ by $2x + 4x^3 + 3x^2 + 1$.

27. $5a^4 + 2a^2b^2 + ab^3 - 3a^3b$ by $5a^3b - 2ab^3 + 3a^2b^2 + b^4$.

28. $4a^7y - 32ay^4 - 8a^5y^2 + 16a^3y^3$ by $a^6y^2 + 4a^2y^4 + 4a^4y^3$.

29. $3m^3 + 3n^3 + 9mn^2 + 9m^2n$ by $6m^2n^3 - 2mn^4 - 6m^3n^2 + 2m^4n$.

30. $6a^5b + 3a^2b^4 - 2ab^5 + b^6$ by $4a^4 - 2ab^3 - 3b^4$.

Find the products of:

31. $x - 3$, $x - 1$, $x + 1$, and $x + 3$.

32. $x^2 - x + 1$, $x^2 + x + 1$, and $x^4 - x^2 + 1$.

33. $a^2 + ab + b^2$, $a^2 - ab + b^2$, and $a^4 - a^2b^2 + b^4$.

34. $4a^3 - 4a^2b + ab^2$, $4a^2 + 3ab + b^2$, and $2a^2b + b^3$.

35. $x + a$, $x + 2a$, $x - 3a$, $x - 4a$, and $x + 5a$.

36. $9a^2 + b^2$, $27a^3 - b^3$, $27a^3 + b^3$, and $81a^4 - 9a^2b^2 + b^4$.

37. From the product of $y^2 - 2yz - z^2$ and $y^2 + 2yz - z^2$ take the product of $y^2 - yz - 2z^2$ and $y^2 + yz - 2z^2$.

38. Find the dividend when the divisor $= 3a^2 - ab - 3b^2$, the quotient $= a^2b - 2b^3$, the remainder $= -2ab^4 - 6b^5$.

The multiplication of polynomials may be *indicated* by inclosing each in a parenthesis and writing them one after the other. When the operations indicated are actually performed, the expression is said to be *simplified*.

Simplify:

39. $(a+b-c)(a+c-b)(b+c-a)(a+b+c)$.

40. $(a+b)(b+c)-(c+d)(d+a)-(a+c)(b-d)$.

41. $(a+b+c+d)^2+(a-b-c+d)^2 + (a-b+c-d)^2+(a+b-c-d)^2$.

42. $(a+b+c)^2-a(b+c-a)-b(a+c-b)-c(a+b-c)$.

43. $(a-b)x-(b-c)a-\{(b-x)(b-a)-(b-c)(b+c)\}$.

44. $(m+n)m-\{(m-n)^2-(n-m)n\}$.

45. $(a-b+c)^2-\{a(c-a-b)-[b(a+b+c) - c(a-b-c)]\}$.

46. $(p^2+q^2)r-(p+q)(p\{r-q\}-q\{r-p\})$.

47. $(9x^2y^2-4y^4)(x^2-y^2)-\{3xy-2y^2\}\{3x(x^2+y^2) - 2y(y^2+3xy-x^2)\}y$.

48. $a^2-\{2ab-[-(a+\{b-c\})(a-\{b-c\})+2ab] - 4bc\}-(b+c)^2$.

49. $\{ac-(a-b)(b+c)\}-b\{b-(a-c)\}$.

50. $5\{(a-b)x-cy\}-2\{a(x-y)-bx\} - \{3ax-(5c-2a)y\}$.

51. $(x-1)(x-2)-3x(x+3)+2\{(x+2)(x+1)-3\}$.

52. $\{(2a+b)^2+(a-2b)^2\} \times \{(3a-2b)^2-(2a-3b)^2\}.$

53. $4(a-3b)(a+3b)-2(a-6b)^2-2(a^2+6b^2).$

54. $x^2(x^2+y^2)^2-2x^2y^2(x+y)(x-y)-(x^3-y^3)^2.$

55. $16(a^2+b^2)(a^2-b^2)-(2a-3)(2a+3)(4a^2+9)$
$+(2b-3)(2b+3)(4b^2+9).$

---

73. There are some examples in multiplication which occur so often in algebraical operations that they should be carefully noticed and remembered. The three which follow are of great importance:

(1)
$$\begin{array}{l} a+b \\ \underline{a+b} \\ a^2+ab \\ \underline{\qquad ab+b^2} \\ a^2+2ab+b^2 \end{array}$$

(2)
$$\begin{array}{l} a-b \\ \underline{a-b} \\ a^2-ab \\ \underline{\qquad -ab+b^2} \\ a^2-2ab+b^2 \end{array}$$

(3)
$$\begin{array}{l} a+b \\ \underline{a-b} \\ a^2+ab \\ \underline{\qquad -ab-b^2} \\ a^2 \qquad -b^2 \end{array}$$

From (1) we have $(a+b)^2=a^2+2ab+b^2$. That is,

74. *The square of the sum of two numbers is equal to the sum of their squares + twice their product.*

From (2) we have $(a-b)^2=a^2-2ab+b^2$. That is,

75. *The square of the difference of two numbers is equal to the sum of their squares − twice their product.*

From (3) we have $(a+b)(a-b)=a^2-b^2$. That is,

76. *The product of the sum and difference of two numbers is equal to the difference of their squares.*

77. A general truth expressed by symbols is called a **formula.**

78. By using the double sign $\pm$, read plus or minus, we may represent (1) and (2) by a single formula; thus,

$$(a \pm b)^2 = a^2 \pm 2ab + b^2;$$

in which expression the upper signs correspond with one another, and the lower with one another.

By remembering these formulas the square of any binomial, or the product of the sum and difference of any two numbers, may be written by inspection; thus:

Exercise XV.

1. $(127)^2 - (123)^2 = (127+123)(127-123)$
$= 250 \times 4 = 1000.$
2. $(29)^2 = (30-1)^2 = 900 - 60 + 1 = 841.$
3. $(53)^2 = (50+3)^2 = 2500 + 300 + 9 = 2809.$
4. $(3x+2y)^2 = 9x^2 + 12xy + 4y^2.$
5. $(2a^2x - 5x^2y)^2 = 4a^4x^2 - 20a^2x^3y + 25x^4y^2.$
6. $(3ab^2c + 2a^2c^2)(3ab^2c - 2a^2c^2) = 9a^2b^4c^2 - 4a^4c^4.$

7. $(x+y)^2 =$	15. $(ab+cd)^2 =$
8. $(y-z)^2 =$	16. $(3mn-4)^2 =$
9. $(2x+1)^2 =$	17. $(12+5x)^2 =$
10. $(2a+5b)^2 =$	18. $(4xy^2 - yz^2)^2 =$
11. $(1-x^2)^2 =$	19. $(3abc - bcd)^2 =$
12. $(3ax - 4x^2)^2 =$	20. $(4x^3 - xy^2)^2 =$
13. $(1-7a)^2 =$	21. $(x+y)(x-y) =$
14. $(5xy+2)^2 =$	22. $(2a+b)(2a-b) =$

23. $(3-x)(3+x)=$

24. $(3ab+2b^2)(3ab-2b^2)=$

25. $(4x^2-3y^2)(4x^2+3y^2)=$

26. $(a^3x^2-by^4)(a^3x^2+by^4)=$

27. $(6xy-5y^2)(6xy+5y^2)=$

28. $(4x^5-1)(4x^5+1)=$

29. $(1+3ab^3)(1-3ab^3)=$

30. $(ax+by)(ax-by)(a^2x^2+b^2y^2)=$

79. Also the square of a trinomial should be carefully noticed.

$$\begin{array}{l}
a+b+c \\
\underline{a+b+c} \\
a^2+ab+ac \\
\qquad\ ab \qquad +b^2+bc \\
\underline{\qquad\qquad ac \qquad +bc+c^2} \\
a^2+2ab+2ac+b^2+2bc+c^2, \\
=a^2+b^2+c^2+2ab+2ac+2bc.
\end{array}$$

It is evident that this result is composed of two sets of numbers:

I. The squares of a, b, and c;

II. Twice the products of a, b, and c taken two and two.

Again,

$$\begin{array}{l}
a-b-c \\
\underline{a-b-c} \\
a^2-ab-ac \\
\quad -ab \qquad +b^2+bc \\
\underline{\qquad\quad -ac \qquad +bc+c^2} \\
a^2-2ab-2ac+b^2+2bc+c^2 \\
=a^2+b^2+c^2-2ab-2ac+2bc.
\end{array}$$

The law of formation is the same as before:

I. The squares of a, b, and c;
II. Twice the products of a, b, and c taken two and two.

The sign of each double product is $+$ or $-$ according as the signs of the factors composing it are *like* or *unlike*.

The same law holds good for the square of expressions containing more than three terms, and may be stated thus:

80. *To the sum of the squares of the several terms add twice the product of each term by each of the terms that follow it.*

By remembering this formula, the square of any polynomial may be written by inspection; thus:

Exercise XVI.

1. $(x+y+z)^2=$	9. $(a^3+b^3+c^3)^2=$
2. $(x-y+z)^2=$	10. $(x^3-y^3-z^3)^2=$
3. $(m+n-p-q)^2=$	11. $(x+2y-3z)^2=$
4. $(x^2+2x-3)^2=$	12. $(x^2-2y^2+5z^2)^2=$
5. $(x^2-6x+7)^2=$	13. $(x^2+2x-2)^2=$
6. $(2x^2-7x+9)^2=$	14. $(x^2-5x+7)^2=$
7. $(x^2+y^2-z^2)^2=$	15. $(2x^2-3x-4)^2=$
8. $(x^4-4x^2y^2+y^4)^2=$	16. $(x+2y+3z)^2=$

81. Likewise, the product of two binomials of the form $x+a$, $x+b$ should be carefully noticed and remembered.

(1)
$$\begin{array}{r}
x+5\\
\underline{x+3}\\
x^2+5x\\
\underline{3x+15}\\
x^2+8x+15
\end{array}$$

(2)
$$\begin{array}{r}
x-5\\
\underline{x-3}\\
x^2-5x\\
\underline{-3x+15}\\
x^2-8x+15
\end{array}$$

(3)
$$\begin{array}{l} x+5 \\ \underline{x-3} \\ x^2+5x \\ \quad\ \underline{-3x-15} \\ x^2+2x-15 \end{array}$$

(4)
$$\begin{array}{l} x-5 \\ \underline{x+3} \\ x^2-5x \\ \quad\ \underline{+3x-15} \\ x^2-2x-15 \end{array}$$

It will be observed that:

I. In all the results the first term is x^2 and the last term is the product of 5 and 3.

II. From (1) and (2), when the second terms of the binomials have *like* signs, the product has

the last term *positive;*

the *coefficient* of the middle term = the sum of 3 and 5;

the *sign* of the middle term is the same as that of the 3 and 5.

III. From (3) and (4), when the second terms of the binomials have *unlike* signs, the product has

the last term *negative;*

the *coefficient* of the middle term = the **difference** of 3 and 5;

the *sign* of the middle term is that of the **greater** of the two numbers.

82. These results may be deduced from the general formula,

$$(x+a)(x+b)=x^2+(a+b)x+ab,$$

by supposing for (1) a and b both positive;

(2) a and b both negative;

(3) a positive, b negative, and $a > b$;

(4) a negative, b positive, and $a > b$.

By remembering this formula the product of two binomials may be written by inspection; thus:

Exercise XVII.

1. $(x+2)(x+3)=$
2. $(x+1)(x+5)=$
3. $(x-3)(x-6)=$
4. $(x-8)(x-1)=$
5. $(x-8)(x+1)=$
6. $(x-2)(x+5)=$
7. $(x-3)(x+7)=$
8. $(x-2)(x-4)=$
9. $(x+1)(x+11)=$
10. $(x-2a)(x+3a)=$
11. $(x-c)(x-d)=$
12. $(x-4y)(x+y)=$
13. $(a-2b)(a-5b)=$
14. $(x^2+2y^2)(x^2+y^2)=$
15. $(x^2-3xy)(x^2+xy)=$
16. $(ax-9)(ax+6)=$
17. $(x+a)(x-b)=$
18. $(x-11)(x+4)=$
19. $(x+12)(x-11)=$
20. $(x-10)(x-5)=$

83. The second, third, and fourth powers of $a+b$ are found in the following manner:

$$\begin{array}{rl}
 & a+b \\
 & \underline{a+b} \\
 & a^2+ab \\
 & \underline{ab+b^2} \\
(a+b)^2= & a^2+2ab+b^2 \\
 & \underline{a+b} \\
 & a^3+2a^2b+ab^2 \\
 & \underline{a^2b+2ab^2+b^3} \\
(a+b)^3= & a^3+3a^2b+3ab^2+b^3 \\
 & \underline{a+b} \\
 & a^4+3a^3b+3a^2b^2+ab^3 \\
 & \underline{a^3b+3a^2b^2+3ab^3+b^4} \\
(a+b)^4= & a^4+4a^3b+6a^2b^2+4ab^3+b^4
\end{array}$$

From these results it will be observed that:

I. The number of terms is greater by one than the exponent of the power to which the binomial is raised.

II. In the first term, the exponent of a is the same as the exponent of the power to which the binomial is raised; and it decreases by one in each succeeding term.

III. b appears in the second term with 1 for an exponent, and its exponent increases by one in each succeeding term.

IV. The coefficient of the first term is 1.

V. The coefficient of the second term is the same as the exponent of the power to which the binomial is raised.

VI. The coefficient of each succeeding term is found from the next preceding term by multiplying its coefficient by the exponent of a, and dividing the product by a number greater by one than the exponent of b.

84. If b be negative, the terms in which the odd powers of b occur are negative. Thus:

$$(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

Exercise XVIII.

Write by inspection the results:

1. $(x+a)^3=$	**5.** $(x+a)^4=$	**9.** $(x+y)^5=$
2. $(x-a)^3=$	**6.** $(x-a)^4=$	**10.** $(x-y)^5=$
3. $(x+1)^3=$	**7.** $(x+1)^4=$	**11.** $(x+1)^5=$
4. $(x-1)^3=$	**8.** $(x-1)^4=$	**12.** $(x-1)^5=$

CHAPTER IV.

DIVISION.

85. Division is the operation by which, when a product and one of its factors are given, the other factor is determined.

86. With reference to this operation the product is called the **dividend**; the given factor the **divisor**; and the required factor the **quotient**.

87. The operation of division is indicated by the sign $\div$; by the colon :, or by writing the dividend over the divisor with a line drawn between them. Thus, $12 \div 4$, $12 : 4$, $\frac{12}{4}$, each means that 12 is to be divided by 4.

88. $+12$ divided by $+4$ gives the quotient $+3$; since only a positive number, $+3$, when multiplied by $+4$, can give the positive product, $+12$. § 61.

$+12$ divided by -4 gives the quotient -3; since only a negative number, -3, when multiplied by -4, can give the positive product, $+12$. § 61.

-12 divided by $+4$ gives the quotient -3; since only a negative number, -3, when multiplied by $+4$, can give the negative product, -12. § 61.

-12 divided by -4 gives the quotient $+3$; since only a positive number, $+3$, when multiplied by -4, can give the negative product, -12. § 61.

(1) $\frac{+12}{+4} = +3.$ (3) $\frac{-12}{+4} = -3.$

ponent of the answer ... $+12$ ... (4) $-12 = +3.$

Exercise XX.

1. $\frac{+ab}{+a} = +b.$
2. $\frac{+ab}{-a} = -b.$
3. $\frac{-ab}{+a} = -b.$
4. $\frac{-ab}{-a} = +b.$
5. $\frac{6mx}{2x} =$
6. $\frac{12a^4}{-3a} =$
7. $\frac{10ab}{2bc} =$
8. $\frac{x^3}{-x^5} =$
9. $\frac{-12am}{-2m} =$
10. $\frac{35abcd}{5bd} =$
11. $\frac{abx}{5aby} =$
12. $\frac{27a^7}{-3a^3} =$
13. $\frac{-3bmx}{4ax^2} =$
14. $\frac{ab^2c^3}{abc} =$
15. $\frac{m^5p^2x^4}{mp^2x^2} =$
16. $\frac{-51abdy^2}{3bdy} =$
17. $\frac{225m^2y}{25my^2} =$
18. $\frac{30x^2y^3}{-5x^3y} =$
19. $\frac{4a^2m^4x^5}{5a^5m^3x} =$
20. $\frac{42x^3y^2z^4}{7xy^2z^3} =$
21. $\frac{-3a^2b^3c^4d^5}{-a^4b^2cd^3} =$
22. $\frac{12am^5n^4p^3q^2}{4m^2n^3p^4q^5} =$
23. $(4a^2bz^3 \times 10a^2b^3z) \div 5a^3b^2z^2 =$
24. $(21x^2y^4z^6 \div 3xy^3z)(-2x^3y^2z) =$
25. $104ab^3x^9 \div (91a^5b^6x^7 \div 7a^4b^4x) =$
26. $(24a^5b^3x \div 3a^2b^2) + (35a^6b^2x^3 \div -5a^3bx) =$
27. $85a^{4m+1} \div 5a^{4m-2} =$
28. $84a^{n-4} \div 12a^2 =$

Of Polynomials by Monomials.

95. The product of $(a+b+c)\times p = ap+bp+cp$.

If the product of two factors be divided by one of the factors, the quotient is the other factor. Therefore,

$$(ap+bp+cp)\div p = a+b+c.$$

But a, b, and c are the quotients obtained by dividing each term, ap, bp, and cp, by p.

Therefore, to divide a polynomial by a monomial,

96. *Divide each term of the polynomial by the monomial.*

Exercise XXI.

1. $(8ab-12ac)\div 4a = 2b-3c.$
2. $(15am-10bm+20cm)\div -5m = -3a+2b-4c.$
3. $(18amy-27bny+36cpy)\div -9y =$
4. $(21ax-18bx+15cx)\div -3x =$
5. $(12x^5-8x^3+4x)\div 4x =$
6. $(3x^3-6x^5+9x^7-12x^9)\div 3x^2 =$
7. $(35m^3y+28m^2y^2-14my^3)\div -7my =$
8. $(4a^4b-6a^3b^2+12a^2b^3)\div 2a^2b =$
9. $(12x^3y^3-15x^4y^2-24x^5y)\div -3x^2y =$
10. $(12x^5y^4-24x^4y^3+36x^3y^3-12x^2y^2)\div 12x^2y^2 =$
11. $(3a^4-2a^5b-a^6b^2)\div a^4 =$
12. $(3x^3yz^2+6x^4yz^3-15x^5y^2z^3+18x^6y^3z)\div -3x^3yz =$
13. $(-16a^3b^2c^5+8a^4b^2c^4-12a^5b^3c^3)\div -4a^2b^2c^2 =$

Of Polynomials by Polyno.

99. It
and divisor ar-

7.

8. by $2x-3$.

9. 6 $-4x+24$ by $2x+6$.

10. $3x^2$ $x+9x^3-1$ by $3x-1$.

11. $7x^3+58x-24x^2-21$ by $7x-3$.

12. x^6-1 by $x-1$.

13. $a^3-2ab^2+b^3$ by $a-b$.

14. x^4-81y^4 by $x-3y$.

15. x^5-y^5 by $x-y$.

16. a^5+32b^5 by $a+2b$.

17. $2a^4+27ab^3-81b^4$ by $a+3b$.

18. $x^4+11x^2-12x-5x^3+6$ by $3+x^2-3x$.

19. $x^4-9x^2+x^3-16x-4$ by x^2+4+4x.

20. $36+x^4-13x^2$ by $6+x^2+5x$.

21. x^4+64 by x^2+4x+8.

22. $x^4+x^3+57-35x-24x^2$ by x^2-3+2x.

23. $1-x-3x^2-x^5$ by $1+2x+x^2$.

24. x^6-2x^3+1 by x^2-2x+1.

25. $a^4+2a^2b^2+9b^4$ by $a^2-2ab+3b^2$.

26. $4x^5-x^3+4x$ by $2+2x^2+3x$.

27. a^5-243 by $a-3$.

28. $18x^4+82x^2+40-67x-45x^3$ by $3x^2+5-4x$.

29. $x^4-6xy-9x^2-y^2$ by x^2+y+3x.

30. $x^4+9x^2y^2-6x^3y-4y^4$ by $x^2-3xy+2y^2$.

31. $x^4+x^2y^2+y^4$ by x^2-xy+y^2.

32. $x^5+x^3+x^4y+y^3-2xy^2-x^3y^2$ by x^2+x-y.

33. $2x^2-3y^2+xy-xz-4yz-z^2$ by $2x+3y+z$.

34. $12+82x^2+106x^4-70x^5-112x^3-38x$ by $3-5x+7x^2$.

35. x^5+y^5 by $x^4-x^3y+x^2y^2-xy^3+y^4$.

36. $2x^4+2x^2y^2-2xy^3-7x^3y-y^4$ by $2x^2+y^2-xy$.

37. $16x^4+4x^2y^2+y^4$ by $4x^2-2xy+y^2$.

38. $32a^5b+8a^3b^3-ab^5-4a^2b^4-56a^4b^2$ by $b^2-4a^2b+6ab^2$.

39. $1+5x^3-6x^4$ by $1-x+3x^2$.

40. $1-52a^4b^4-51a^3b^3$ by $4a^2b^2+3ab-1$.

41. x^7y-xy^7 by $x^3y+2xy^3-2x^2y^2-y^4$.

42. $x^6+15x^4y^2+15x^2y^4+y^6-6x^5y-6xy^5-20x^3y^3$ by $x^3-3x^2y+3xy^2-y^3$.

43. $a^7+2a^3b^4-2a^4b^3-2a^6b-6a^2b^5-3ab^6$ by $a^3-2a^2b-ab^2$.

44. $81x^6y+18x^2y^5-54x^5y^2-18x^3y^4-18xy^6-9y^7$ by $3x^4+x^2y^2+y^4$.

45. $a^4+2a^3b+8a^2b^2+8ab^3+16b^4$ by a^2+4b^2.

46. $8y^6-x^6+21x^3y^3-24xy^5$ by $3xy-x^2-y^2$.

47. $16a^4+9b^4+8a^2b^2$ by $4a^2+3b^2-4ab$.

48. $a^3+b^3+c^3-3abc$ by $a+b+c$.

49. $a^3+8b^3+c^3-6abc$ by $a^2+4b^2+c^2-ac-2ab-2bc$.

50. $a^3+b^3+c^3+3a^2b+3ab^2$ by $a+b+c$.

100. The operation of division may be shortened in some cases by the use of parentheses. Thus:

$$
\begin{array}{l|l}
x^3+(a+b+c)x^2+(ab+ac+bc)x+abc & x+b \\ \cline{2-2}
\underline{x^3+(\quad+b\quad)x^2\qquad\qquad\qquad\quad} & x^2+(a+c)x+ac \\
\quad(a\quad+c)x^2+(ab+ac+bc)x & \\
\quad\underline{(a\quad+c)x^2+(ab\qquad+bc)x\quad} & \\
\qquad\qquad\qquad\qquad acx\qquad+abc & \\
\qquad\qquad\qquad\qquad \underline{acx\qquad+abc} &
\end{array}
$$

Exercise XXIII.

Divide

1. $a^2(b+c)+b^2(a-c)+c^2(a-b)+abc$ by $a+b+c$.
2. $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$ by $x^2-(a+b)x+ab$.
3. $x^3-2ax^2+(a^2+ab-b^2)x-a^2b+ab^2$ by $x-a+b$.
4. $x^4-(a^2-b-c)x^2-(b-c)ax+bc$ by x^2-ax+c.
5. $y^3-(m+n+p)y^2+(mn+mp+np)y-mnp$ by $y-p$.
6. $x^4+(5+a)x^3-(4-5a+b)x^2-(4a+5b)x+4b$ by x^2+5x-4.
7. $x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$ $-(abc+abd+acd+bcd)x+abcd$ by $x^2-(a+c)x+ac$.
8. $x^5-(m-c)x^4+(n-cm+d)x^3+(r+cn-dm)x^2$ $+(cr+dn)x+dr$ by x^3-mx^2+nx+r.
9. $x^5-mx^4+nx^3-nx^2+mx-1$ by $x-1$.
10. $(x+y)^3+3(x+y)^2z+3(x+y)z^2+z^3$ by $(x+y)^2+2(x+y)z+z^2$.

101. There are some cases in Division which occur so often in algebraic operations that they should be carefully noticed and remembered.

CASE I.

The student may easily verify the following results:

(1) $\dfrac{a^3-b^3}{a-b}=a^2+ab+b^2.$

(2) $\dfrac{27a^3-8b^3}{3a-2b}=9a^2+6ab+4b^2.$

(3) $\dfrac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4.$

(4) $\dfrac{a^5-32b^5}{a-2b}=a^4+2a^3b+4a^2b^2+8ab^3+16b^4.$

From these results it may be assumed that:

102. *The difference of two equal odd powers of any two numbers is divisible by the difference of the numbers.*

It will also be seen that:

I. The number of terms in the quotient is equal to the exponent of the powers.

II. The signs of the quotient are all positive.

III. The first term of the quotient is obtained, as usual, by dividing the first term of the dividend by the first term of the divisor.

IV. Each succeeding term of the quotient may be obtained by dividing the preceding term of the quotient by the first term of the divisor, and multiplying the result by the second term of the divisor (disregarding the sign).

EXERCISE XXIV.

Write by inspection the results in the following examples:

1. $(y^3-1)\div(y-1)$.
2. $(b^3-125)\div(b-5)$.
3. $(a^3-216)\div(a-6)$.
4. $(x^3-343)\div(x-7)$.
5. $(x^5-y^5)\div(x-y)$.
6. $(a^5-1)\div(a-1)$.
7. $(1-8x^3)\div(1-2x)$.
8. $(x^5-32b^5)\div(x-2b)$.
9. $(8a^3x^3-1)\div(2ax-1)$.
10. $(1-27x^3y^3)\div(1-3xy)$.
11. $(64a^3b^3-27x^3)\div(4ab-3x)$.
12. $(243a^5-1)\div(3a-1)$.
13. $(32a^5-243b^5)\div(2a-3b)$.

CASE II.

(1) $\dfrac{a^3+b^3}{a+b}=a^2-ab+b^2.$

(2) $\dfrac{27x^3+8y^3}{3x+2y}=9x^2-6xy+4y^2.$

(3) $\dfrac{a^5+b^5}{a+b}=a^4-a^3b+a^2b^2-ab^3+b^4.$

(4) $\dfrac{243x^5+32y^5}{3x+2y}=81x^4-54x^3y+36x^2y^2-24xy^3+16y^4.$

From these results it may be assumed that:

103. *The sum of two equal odd powers of two numbers is divisible by the sum of the numbers.*

The quotient may be found as in Case I., but the signs are alternately plus and minus.

EXERCISE XXV.

Write by inspection the results in the following examples:

1. $(x^3+y^3)\div(x+y)$.
2. $(x^5+y^5)\div(x+y)$.
3. $(1+8a^3)\div(1+2a)$.
4. $(27a^3+b^3)\div(3a+b)$.
5. $(8a^3x^3+1)\div(2ax+1)$.
6. $(x^3+27y^3)\div(x+3y)$.
7. $(a^5+32b^5)\div(a+2b)$.
8. $(512x^3y^3+z^3)\div(8xy+z)$.
9. $(729a^3+216b^3)\div(9a+6b)$.
10. $(64a^3+1000b^3)\div(4a+10b)$.
11. $(64a^3b^3+27x^3)\div(4ab+3x)$.
12. $(x^3+343)\div(x+7)$.
13. $(27x^3y^3+8z^3)\div(3xy+2z)$.
14. $(1024a^5+243b^5)\div(4a+3b)$.

CASE III.

(1) $\dfrac{x^2-y^2}{x-y}=x+y$.

(2) $\dfrac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3$.

(3) $\dfrac{x^2-y^2}{x+y}=x-y$.

(4) $\dfrac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3$.

From these results it may be assumed that:

104. *The difference of two equal even powers of two numbers is divisible by the difference and also by the sum of the numbers.*

When the divisor is the difference of the numbers, the quotient is found as in Case I.

When the divisor is the sum of the numbers, the quotient is found as in Case II.

EXERCISE XXVI.

Write by inspection the results in the following examples:

1. $(x^4 - y^4) \div (x - y)$.
2. $(x^4 - y^4) \div (x + y)$.
3. $(a^6 - x^6) \div (a - x)$.
4. $(a^6 - x^6) \div (a + x)$.
5. $(x^4 - 81y^4) \div (x - 3y)$.
6. $(x^4 - 81y^4) \div (x + 3y)$.
7. $(16x^4 - 1) \div (2x - 1)$.
8. $(16x^4 - 1) \div (2x + 1)$.
9. $(81a^4x^4 - 1) \div (3ax - 1)$.
10. $(81a^4x^4 - 1) \div (3ax + 1)$.
11. $(64a^6 - b^6) \div (2a - b)$.
12. $(64a^6 - b^6) \div (2a + b)$.
13. $(x^6 - 729y^6) \div (x - 3y)$.
14. $(x^6 - 729y^6) \div (x + 3y)$.
15. $(81a^4 - 16c^4) \div (3a - 2c)$.
16. $(81a^4 - 16c^4) \div (3a + 2c)$.
17. $(256a^4 - 10{,}000) \div (4a - 10)$.
18. $(256a^4 - 10{,}000) \div (4a + 10)$.
19. $(625x^4 - 1) \div (5x - 1)$.

CASE IV.

It may be easily verified that:

105. *The sum of two equal even powers of two numbers is not divisible by either the sum or the difference of the numbers.*

But when the exponent of each of the two equal powers is composed of an *odd* and an *even* factor, the sum of the given powers is divisible by the sum of the powers expressed by the even factor.

Thus, $x^6 + y^6$ is not divisible by $x + y$ or by $x - y$, but is divisible by $x^2 + y^2$.

The quotient may be found as in Case II.

Exercise XXVII.

Write by inspection the results in the following examples:

1. $(x^6+y^6)\div(x^2+y^2)$.
2. $(a^6+1)\div(a^2+1)$.
3. $(a^{10}+y^{10})\div(a^2+y^2)$.
4. $(b^{10}+1)\div(b^2+1)$.
5. $(a^{12}+b^{12})\div(a^4+b^4)$.
6. $(x^{12}+1)\div(x^4+1)$.
7. $(64x^6+y^6)\div(4x^2+y^2)$.
8. $(64+a^6)\div(4+a^2)$.
9. $(729a^6+b^6)\div(9a^2+b^2)$.
10. $(729c^6+1)\div(9c^2+1)$.

Note. The introduction of negative numbers requires an extension of the meaning of some terms common to arithmetic and algebra. But every such extension of meaning must be consistent with the sense previously attached to the term and with general laws already established.

Addition in algebra does not necessarily imply *augmentation*, as it does in arithmetic. Thus, $7+(-5)=2$. The word **sum**, however, is used to denote the result.

Such a result is called the **algebraic sum**, when it is necessary to distinguish it from the *arithmetical sum*, which would be obtained by adding the *absolute values* of the numbers.

The general definition of Addition is, the operation of uniting two or more numbers in a *single expression* written in its simplest form.

The general definition of Subtraction is, the operation of finding from two given numbers, called *minuend* and *subtrahend*, a third number, called *difference*, which added to the subtrahend will give the minuend.

The general definition of Multiplication is, the operation of finding from two given numbers, called *multiplicand* and *multiplier*, a third number, called *product*, which may be formed from the multiplicand as the multiplier is formed from unity.

The general definition of Division is, the operation of finding the *other factor* when the *product* of two factors and *one factor* are given.

CHAPTER V.

SIMPLE EQUATIONS.

106. An **equation** is a statement that two expressions are equal. Thus, $4x - 12 = 8$.

107. Every equation consists of two parts, called the first and second *sides*, or *members*, of the equation.

108. An **identical equation** is one in which the two sides are equal, whatever numbers the letters stand for. Thus, $(x+b)(x-b) = x^2 - b^2$.

109. An **equation of condition** is one which is true only when the letters stand for particular values. Thus, $x + 5 = 8$ is true only when $x = 3$.

110. A letter to which a particular value must be given in order that the statement contained in an equation may be true is called an *unknown quantity*.

111. The *value* of the unknown quantity is the number which substituted for it will *satisfy* the equation, and is called a *root* of the equation.

112. To **solve** an equation is to find the value of the unknown quantity.

113. A **simple equation** is one which contains only the *first* power of the unknown quantity, and is also called an equation of the *first degree*.

114. *If equal changes be made in both sides of an equation, the results will be equal.* § 43.

(1) To find the value of x in $x+b=a$.

$$x+b=a;$$

Subtract b from each side, $x+b-b=a-b$;

Cancel $+b-b$, $x=a-b$.

(2) To find the value of x in $x-b=a$.

$$x-b=a;$$

Subtract $-b$ from each side, $x-b+b=a+b$;

Cancel $-b+b$, $x=a+b$.

The result in each case is the same as if b were transposed to the other side of the equation with its sign changed. Therefore,

115. *Any term may be transposed from one side of an equation to the other provided its sign be changed.*

For, in this transposition, the same number is subtracted from each side of the equation.

116. The signs of all the terms on each side of an equation may be changed; for, this is in effect transposing every term.

117. When the known and unknown quantities of an equation are connected by the sign $+$ or $-$, they may be separated by transposing the known quantities to one side and the unknown to the other.

118. Hence, to solve an equation with one unknown quantity,

Transpose all the terms involving the unknown quantity to the left side, and all the other terms to the right side:

combine the like terms, and divide both sides by the coefficient of the unknown quantity.

119. To verify the result, substitute the value of the unknown quantity in the original equation.

Exercise XXVIII.

Find the value of x in

1. $5x-1=19$.
2. $3x+6=12$.
3. $24x=7x+34$.
4. $8x-29=26-3x$.
5. $12-5x=19-12x$.
6. $3x+6-2x=7x$.
7. $5x+50=4x+56$.
8. $16x-11=7x+70$.
9. $24x-49=19x-14$.
10. $3x+23=78-2x$.
11. $26-8x=80-14x$.
12. $13-3x=5x-3$.
13. $3x-22=7x+6$.
14. $8+4x=12x-16$.
15. $5x-(3x-7)=4x-(6x-35)$.
16. $6x-2(9-4x)+3(5x-7)=10x-(4+16x+35)$.
17. $9x-3(5x-6)+30=0$.
18. $x-7(4x-11)=14(x-5)-19(8-x)-61$.
19. $(x+7)(x-3)=(x-5)(x-15)$.
20. $(x-8)(x+12)=(x+1)(x-6)$.
21. $(x-2)(7-x)+(x-5)(x+3)-2(x-1)+12=0$.
22. $(2x-7)(x+5)=(9-2x)(4-x)+229$.
23. $14-x-5(x-3)(x+2)+(5-x)(4-5x)=45x-76$.
24. $(x+5)^2-(4-x)^2=21x$.
25. $5(x-2)^2+7(x-3)^2=(3x-7)(4x-19)+42$.

EXERCISE XXIX.

PROBLEMS.

1. Find a number such that when 12 is added to its double the sum shall be 28.

Let $x =$ the number.
Then $2x =$ its double,
and $2x + 12 =$ double the number increased by 12.
But $28 =$ double the number increased by 12.

$$\therefore 2x + 12 = 28.$$
$$2x = 28 - 12,$$
$$2x = 16,$$
$$x = 8.$$

2. A farmer had two flocks of sheep, each containing the same number. He sold 21 sheep from one flock and 70 from the other, and then found that he had left in one flock twice as many as in the other. How many had he in each?

Let $x =$ number of sheep in each flock.
Then $x - 21 =$ number of sheep left in one flock,
and $x - 70 =$ number of sheep left in the other.

$$\therefore x - 21 = 2(x - 70),$$
$$x - 21 = 2x - 140.$$
$$x - 2x = -140 + 21,$$
$$-x = -119,$$
$$x = 119.$$

3. A and B had equal sums of money; B gave A \$5, and then 3 times A's money was equal to 11 times B's money. What had each at first?

Let $x =$ number of dollars each had.
Then $x + 5 =$ number of dollars A had after receiving \$5 from B,
and $x - 5 =$ number of dollars B had after giving A \$5.

$$\therefore 3(x+5)=11(x-5),$$
$$3x+15=11x-55.$$
$$3x-11x=-55-15.$$
$$-8x=-70,$$
$$x=8\tfrac{3}{4}.$$

Therefore, each had \$8.75.

4. Find a number whose treble exceeds 50 by as much as its double falls short of 40.

Let $x=$ the number.
Then $3x=$ its treble,
and $3x-50=$ the excess of its treble over 50;
also, $40-2x=$ the number its double lacks of 40.

$$\therefore 3x-50=40-2x;$$
$$3x+2x=40+50.$$
$$5x=90,$$
$$x=18.$$

5. What two numbers are those whose difference is 14, and whose sum is 48?

Let $x=$ the larger number.
Then $48-x=$ the smaller number,
and $x-(48-x)=$ the difference of the numbers.
But $14=$ the difference of the numbers.

$$\therefore x-(48-x)=14.$$
$$x-48+x=14;$$
$$2x=62;$$
$$x=31.$$

Therefore, the two numbers are 31 and 17.

6. To the double of a certain number I add 14, and obtain as a result 154. What is the number?

7. To four times a certain number I add 16, and obtain as a result 188. What is the number?

8. By adding 46 to a certain number, I obtain as a result a number three times as large as the original number. Find the original number.

9. One number is three times as large as another. If I take the smaller from 16 and the greater from 30, the remainders are equal. What are the numbers?

10. Divide the number 92 into four parts, such that the first exceeds the second by 10, the third by 18, and the fourth by 24.

11. The sum of two numbers is 20; and if three times the smaller number be added to five times the greater, the sum is 84. What are the numbers?

12. The joint ages of a father and son are 80 years. If the age of the son were doubled, he would be 10 years older than his father. What is the age of each?

13. A man has 6 sons, each 4 years older than the next younger. The eldest is three times as old as the youngest. What is the age of each?

14. Add $24 to a certain sum and the amount will be as much above $80 as the sum is below $80. What is the sum?

15. Thirty yards of cloth and 40 yards of silk together cost $330; and the silk cost twice as much a yard as the cloth. How much does each cost a yard?

16. Find the number whose double increased by 24 exceeds 80 by as much as the number itself is less than 100.

17. The sum of $500 is divided among A, B, C, and D. A and B have together $280, A and C $260, and A and D $220. How much does each receive?

18. In a company of 266 persons composed of men, women, and children, there are twice as many men as women, and twice as many women as children. How many are there of each?

19. Find two numbers differing by 8, such that four times the less may exceed twice the greater by 10.

20. A is 58 years older than B, and A's age is as much above 60 as B's age is below 50. Find the age of each.

21. A man leaves his property, amounting to $7500, to be divided among his wife, his two sons, and three daughters, as follows: a son is to have twice as much as a daughter, and the wife $500 more than all the children together. How much was the share of each?

22. A vessel containing some water was filled by pouring in 42 gallons, and there was then in the vessel seven times as much as at first. How much did the vessel hold?

23. A has $72 and B has $52. B gives A a certain sum; then A has three times as much as B. How much did A receive from B?

24. Divide 90 into two such parts that four times one part may be equal to five times the other.

25. Divide 60 into two such parts that one part exceeds the other by 24.

26. Divide 84 into two such parts that one part may be less than the other by 36.

NOTE I. When we have to compare the ages of two persons at a given time, and also a number of years after or before the given time, we must remember that *both* persons will be so many years older or younger.

Thus, if x represent A's age, and $2x$ B's age, at the present time, A's age five years ago will be represented by $x-5$; and B's by $2x-5$. A's age five years hence will be represented by $x+5$; and B's age by $2x+5$.

27. A is twice as old as B, and 22 years ago he was three times as old as B. What is A's age?

28. A father is 30 and his son 6 years old. In how many years will the father be just twice as old as the son?

29. A is twice as old as B, and 20 years since he was three times as old. What is B's age?

30. A is three times as old as B, and 19 years hence he will be only twice as old as B. What is the age of each?

31. A man has three nephews; his age is 50, and the joint ages of the nephews is 42. How long will it be before the joint ages of the nephews will be equal to that of the uncle?

NOTE II. In problems involving quantities of the same kind expressed in different units, we must be careful to reduce all the quantities to the *same unit.*

Thus, if x denote a number of inches, all the quantities of the same kind involved in the problem must be reduced to inches.

32. A sum of money consists of dollars and twenty-five-cent pieces, and amounts to $20. The number of coins is 50. How many are there of each sort?

33. A person bought 30 pounds of sugar of two different kinds, and paid for the whole $2.94. The better kind cost 10 cents a pound and the poorer kind 7 cents a pound. How many pounds were there of each kind?

34. A workman was hired for 40 days, at $1 for every day he worked, but with the condition that for every day he did not work he was to pay 45 cents for his board. At the end of the time he received $22.60. How many days did he work?

35. A wine merchant has two kinds of wine; one worth 50 cents a quart, and the other 75 cents a quart. From these he wishes to make a mixture of 100 gallons, worth $2.40 a gallon. How many gallons must he take of each kind.

36. A gentleman gave some children 10 cents each, and had a dollar left. He found that he would have required one dollar more to enable him to give them 15 cents each. How many children were there?

37. Two casks contain equal quantities of vinegar; from the first cask 34 quarts are drawn, from the second, 20 gallons; the quantity remaining in one vessel is now twice that in the other. How much did each cask contain at first?

38. A gentleman hired a man for 12 months, at the wages of $90 and a suit of clothes. At the end of 7 months the man quits his service and receives $33.75 and the suit of clothes. What was the price of the suit of clothes?

39. A man has three times as many quarters as half-dollars, four times as many dimes as quarters, and twice as many half-dimes as dimes. The whole sum is $7.30. How many coins has he altogether?

40. A person paid a bill of $15.25 with quarters and half-dollars, and gave 51 pieces of money altogether. How many of each kind were there?

41. A bill of 100 pounds was paid with guineas (21 shillings) and half-crowns ($2\frac{1}{2}$ shillings), and 48 more half-crowns than guineas were used. How many of each were paid?

CHAPTER VI.

FACTORS.

120. In multiplication we determine the *product* of two given factors; it is often important to determine the *factors* of a given product.

121. CASE I. The simplest case is that in which all the terms of an expression have one common factor. Thus,

(1) $x^2 + xy = x(x + y)$.

(2) $6a^3 + 4a^2 + 8a = 2a(3a^2 + 2a + 4)$.

(3) $18a^3b - 27a^2b^2 + 36ab = 9ab(2a^2 - 3ab + 4)$.

EXERCISE XXX.

Resolve into factors:

1. $5a^2 - 15a$.
2. $6a^3 + 18a^2 - 12a$.
3. $49x^2 - 21x + 14$.
4. $4x^3y - 12x^2y^2 + 8xy^3$.
5. $y^4 - ay^3 + by^2 + cy$.
6. $6a^5b^3 - 21a^4b^2 + 27a^3b^4$.
7. $54x^2y^6 + 108x^4y^8 - 243x^6y^9$.
8. $45x^7y^{10} - 90x^5y^7 - 360x^4y^8$.
9. $70a^3y^4 - 140a^2y^5 + 210ay^6$.
10. $32a^3b^6 + 96a^6b^8 - 128a^8b^9$.

122. CASE II. Frequently the terms of an expression can be so arranged as to show a common factor. Thus,

$$(1)\quad \begin{aligned} x^2+ax+bx+ab &= (x^2+ax)+(bx+ab), \\ &= x(x+a)+b(x+a), \\ &= (x+b)(x+a). \end{aligned}$$

$$(2)\quad \begin{aligned} ac-ad-bc+bd &= (ac-ad)-(bc-bd), \\ &= a(c-d)-b(c-d), \\ &= (a-b)(c-d). \end{aligned}$$

EXERCISE XXXI.

Resolve into factors:

1. $x^2-ax-bx+ab.$
2. $ab+ay-by-y^2.$
3. $bc+bx-cx-x^2.$
4. $mx+mn+ax+an.$
5. $cdx^2-cxy+dxy-y^2.$
6. $abx-aby+pqx-pqy.$
7. $cdx^2+adxy-bcxy-aby^2.$
8. $abcy-b^2dy-acdx+bd^2x.$
9. $ax-ay-bx+by.$
10. $cdz^2-cyz+dyz-y^2.$

123. The square root of a number is one of the *two equal* factors of that number. Thus, the square root of 25 is 5; for, $25=5\times 5$.

The square root of a^4 is a^2; for, $a^4=a^2\times a^2$.

The square root of $a^2b^2c^2$ is abc; for, $a^2b^2c^2=abc\times abc$.

In general, the square root of a power of a number is expressed by writing the number with an exponent equal to one-half the exponent of the power.

The square root of a product may be found by taking the square root of each factor, and finding the product of the roots.

The square root of a positive number may be either positive or negative; for,

$$a^2 = a \times a,$$

or,

$$a^2 = -a \times -a;$$

but throughout this chapter only the positive value of the square root will be taken.

124. CASE III. From § 73 it is seen that a trinomial is often the product of two binomials. Conversely, a trinomial may, in certain cases, be resolved into two binomial factors. Thus,

To find the factors of

$$x^2 + 7x + 12.$$

The first term of each binomial factor will obviously be x.

The second terms of the two binomial factors must be two numbers

whose *product* is 12,

and whose *sum* is 7.

The only two numbers whose product is 12 and whose sum is 7 are 4 and 3.

$$\therefore x^2 + 7x + 12 = (x + 4)(x + 3).$$

Again, to find the factors of $x^2 + 5xy + 6y^2$.

The first term of each binomial factor will obviously be x.

The second terms of the two binomial factors must be two numbers

whose *product* is $6y^2$,

and whose *sum* is $5y$.

The only two numbers whose product is $6y^2$ and whose sum is $5y$ are $3y$ and $2y$.

$$\therefore x^2 + 5xy + 6y^2 = (x + 3y)(x + 2y).$$

Exercise XXXII.

Find the factors of:

1. $x^2+11x+24.$
2. $x^2+11x+30.$
3. $y^2+17y+60.$
4. $z^2+13z+12.$
5. $x^2+21x+110.$
6. $y^2+35y+300.$
7. $b^2+23b+102.$
8. $x^2+3x+2.$
9. $x^2+7x+6.$
10. $a^2+9ab+8b^2.$
11. $x^2+13ax+36a^2.$
12. $y^2+19py+48p^2.$
13. $z^2+29qz+100q^2.$
14. $a^4+5a^2+6.$
15. $z^6+4z^3+3.$
16. $a^2b^2+18ab+32.$
17. $x^8y^4+7x^4y^2+12.$
18. $z^{10}+10z^5+16.$
19. $a^2+9ab+20b^2.$
20. $x^6+9x^3+20.$
21. $a^2x^2+14abx+33b^2.$
22. $a^2c^2+7acx+10x^2.$
23. $x^2y^2z^2+19xyz+48.$
24. $b^2c^2+18abc+65a^2.$
25. $r^2s^2+23rsz+90z^2.$
26. $m^4n^4+20m^2n^2pq+51p^2q^2.$

125. Case IV. To find the factors of

$$x^2-9x+20.$$

The second terms of the two binomial factors must be two numbers

whose *product* is 20,

and whose *sum* is -9.

The only two numbers whose product is 20 and whose sum is -9 are -5 and -4.

$$\therefore x^2-9x+20=(x-5)(x-4).$$

Exercise XXXIII.

Resolve into factors:

1. $x^2 - 7x + 10$.
2. $x^2 - 29x + 190$.
3. $a^2 - 23a + 132$.
4. $b^2 - 30b + 200$.
5. $z^2 - 43z + 460$.
6. $x^2 - 7x + 6$.
7. $x^4 - 4a^2x^2 + 3a^4$.
8. $x^2 - 8x + 12$.
9. $z^2 - 57z + 56$.
10. $y^6 - 7y^3 + 12$.
11. $x^2y^2 - 27xy + 26$.
12. $a^4b^6 - 11a^2b^3 + 30$.
13. $a^2b^2c^2 - 13abc + 22$.
14. $x^2 - 15x + 50$.
15. $x^2 - 20x + 100$.
16. $a^2x^2 - 21ax + 54$.
17. $a^2x^2 - 16abx + 39b^2$.
18. $a^2c^2 - 24acz + 143z^2$.
19. $x^2 - 20x + 91$.
20. $x^2 - 23x + 120$.
21. $z^2 - 53z + 360$.
22. $x^2 - (a + c)x + ac$.
23. $y^2z^2 - 28abyz + 187a^2b^2$.
24. $c^2d^2 - 30abcd + 221a^2b^2$.

126. Case V. To find the factors of

$$x^2 + 2x - 3.$$

The second terms of the two binomial factors must be two numbers

whose *product* is -3,

and whose *sum* is $+2$.

The only two numbers whose product is -3 and whose sum is $+2$ are $+3$ and -1.

$$\therefore x^2 + 2x - 3 = (x + 3)(x - 1).$$

Exercise XXXIV.

Resolve into factors:

1. x^2+6x-7.
2. $x^2+5x-84$.
3. $y^2+7y-60$.
4. $y^2+12y-45$.
5. $z^2+11z-12$.
6. $z^2+13z-140$.
7. $a^2+13a-300$.
8. $a^2+25a-150$.
9. b^8+3b^4-4.
10. $b^2c^2+3bc-154$.
11. $c^{10}+15c^5-100$.
12. $c^2+17c-390$.
13. $a^2+a-132$.
14. $x^2y^2z^2+9xyz-22$.

127. Case VI. To find the factors of

$$x^2-5x-66.$$

The second terms of the two binomial factors must be two numbers whose *product* is -66,
and whose *sum* is -5.

The only two numbers whose product is -66 and whose sum is -5 are -11 and $+6$.

$$\therefore x^2-5x-66=(x-11)(x+6).$$

Exercise XXXV.

Resolve into factors:

1. $x^2-3x-28$.
2. $y^2-7y-18$.
3. $x^2-9x-36$.
4. $z^2-11z-60$.
5. $z^2-13z-14$.
6. $a^2-15a-100$.
7. $c^{10}-9c^5-10$.
8. $x^2-8x-20$.
9. $y^2-5ay-50a^2$.
10. $a^2b^2-3ab-4$.

11. $a^2x^2 - 3ax - 54$.
12. $c^2d^2 - 24cd - 180$.
13. $a^6c^2 - a^3c - 2$.
14. $y^8z^4 - 5y^4z^2 - 84$.
15. $a^2b^2 - 16ab - 36$.
16. $x^2 - (a-b)x - ab$.

We now proceed to the consideration of trinomials which are perfect squares. These are only particular forms of Cases III. and IV., but from their importance demand special attention.

128. Case VII. To find the factors of

$$x^2 + 18x + 81.$$

The second terms of the two binomial factors must be two numbers whose *product* is 81,
and whose *sum* is 18.

The only two numbers whose product is 81 and whose sum is 18 are 9 and 9.

$$\therefore x^2 + 18x + 81 = (x+9)(x+9) = (x+9)^2.$$

Exercise XXXVI.

Resolve into factors:

1. $x^2 + 12x + 36$.
2. $x^2 + 28x + 196$.
3. $x^2 + 34x + 289$.
4. $z^2 + 2z + 1$.
5. $y^2 + 200y + 10{,}000$.
6. $z^4 + 14z^2 + 49$.
7. $x^2 + 36xy + 324y^2$.
8. $y^4 + 16y^2z^2 + 64z^4$.
9. $y^6 + 24y^3 + 144$.
10. $x^2z^2 + 162xz + 6561$.
11. $4a^2 + 12ab^2 + 9b^4$.
12. $9x^2y^4 + 30xy^2z + 25z^2$.
13. $9x^2 + 12xy + 4y^2$.
14. $4a^4x^2 + 20a^2x^3y + 25x^4y^2$.

129. CASE VIII. To find the factors of

$$x^2 - 18x + 81.$$

The second terms of the two trinomials must be two numbers

whose *product* is 81,

and whose *sum* is -18.

The only two numbers whose product is 81 and whose sum is -18 are -9 and -9.

$$\therefore x^2 - 18x + 81 = (x-9)(x-9) = (x-9)^2.$$

EXERCISE XXXVII.

1. $a^2 - 8a + 16.$
2. $a^2 - 30a + 225.$
3. $x^2 - 38x + 361.$
4. $x^2 - 40x + 400.$
5. $y^2 - 100y + 2500.$
6. $y^4 - 20y^2 + 100.$
7. $y^2 - 50yz + 625z^2.$
8. $x^4 - 32x^2y^2 + 256y^4.$
9. $z^6 - 34z^3 + 289.$
10. $4x^4y^2 - 20x^2y^3z + 25y^4z^2.$
11. $16x^2y^4 - 8xy^3z^2 + y^2z^4.$
12. $9a^2b^2c^2 - 6ab^2c^2d + b^2c^2d^2.$
13. $16x^6 - 8x^4y^2 + x^2y^4.$
14. $a^6x^4 - 2a^3bx^2y^4 + b^2y^8.$
15. $36x^2y^2 - 60xy^3 + 25y^4.$
16. $1 - 6ab^3 + 9a^2b^6.$
17. $9m^2n^2 - 24mn + 16.$
18. $4b^2x^2 - 12bx^2y + 9x^2y^2.$
19. $49a^2 - 112ab + 64b^2.$
20. $64x^4y^6 - 160x^4y^3z + 100x^4z^2.$
21. $49a^2b^2c^2 - 28abcx + 4x^2.$
22. $121x^4 - 286x^2y + 169y^2.$
23. $289x^2y^2z^2 - 102xy^2z^2d + 9y^2z^2d^2.$
24. $361x^2y^2z^2 - 76abcxyz + 4a^2b^2c^2.$

130. CASE IX. An expression in the form of two squares, with the negative sign between them, is the product of two factors which may be determined as follows:

Take the square root of the first number, and the square root of the second number.

The *sum* of these roots will form the first factor;

The *difference* of these roots will form the second factor. Thus:

(1) $a^2 - b^2 = (a+b)(a-b).$

(2) $a^2 - (b-c)^2 = \{a + (b-c)\}\{a - (b-c)\},$
$\qquad = \{a + b - c\}\{a - b + c\}.$

(3) $(a-b)^2 - (c-d)^2 = \{(a-b)+(c-d)\}\{(a-b)-(c-d)\},$
$\qquad = \{a-b+c-d\}\{a-b-c+d\}.$

131. The terms of an expression may often be arranged so as to form two squares with the negative sign between them, and the expression can then be resolved into factors. Thus:

$$\begin{aligned} &a^2 + b^2 - c^2 - d^2 + 2ab + 2cd, \\ &= a^2 + 2ab + b^2 - c^2 + 2cd - d^2, \\ &= (a^2 + 2ab + b^2) - (c^2 - 2cd + d^2), \\ &= (a+b)^2 - (c-d)^2, \\ &= \{(a+b) + (c-d)\}\{(a+b) - (c-d)\}, \\ &= \{a + b + c - d\}\{a + b - c + d\}. \end{aligned}$$

132. An expression may often be resolved into three or more factors. Thus:

(1) $$\begin{aligned} x^{16} - y^{16} &= (x^8 + y^8)(x^8 - y^8) \\ &= (x^8 + y^8)(x^4 + y^4)(x^4 - y^4) \\ &= (x^8 + y^8)(x^4 + y^4)(x^2 + y^2)(x^2 - y^2) \\ &= (x^8 + y^8)(x^4 + y^4)(x^2 + y^2)(x + y)(x - y). \end{aligned}$$

(2) $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2,$

$$=\{2(ab+cd)+(a^2+b^2-c^2-d^2)\}\{2(ab+cd)-(a^2+b^2-c^2-d^2)\},$$

$$=\{2ab+2cd+a^2+b^2-c^2-d^2\}\{2ab+2cd-a^2-b^2+c^2+d^2\},$$

$$=\{a^2+2ab+b^2)-(c^2-2cd+d^2)\}\{(c^2+2cd+d^2)-(a^2-2ab+b^2)\},$$

$$=\{(a+b)^2-(c-d)^2\}\{(c+d)^2-(a-b)^2\},$$

$$=\{a+b+(c-d)\}\{a+b-(c-d)\}\{c+d+(a-b)\}\{c+d-(a-b)\},$$

$$=\{a+b+c-d\}\{a+b-c+d\}\{c+d+a-b\}\{c+d-a+b\}.$$

Exercise XXXVIII.

Resolve into factors:

1. $a^2-b^2.$
2. $a^2-16.$
3. $4a^2-25.$
4. $a^4-b^4.$
5. $a^4-1.$
6. $a^8-b^8.$
7. $a^8-1.$
8. $36x^2-49y^2.$
9. $100x^2y^2-121a^2b^2.$
10. $1-49x^2.$
11. $a^4-25b^2.$
12. $(a-b)^2-c^2.$
13. $x^2-(a-b)^2.$
14. $(a+b)^2-(c+d)^2.$
15. $(x+y)^2-(x-y)^2.$
16. $2ab-a^2-b^2+1.$
17. $x^2-2yz-y^2-z^2.$
18. $x^2-2xy+y^2-z^2.$
19. $a^2+12bc-4b^2-9c^2.$
20. $a^2-2ay+y^2-x^2-2xz-z^2.$
21. $2xy-x^2-y^2+z^2.$
22. $x^2+y^2-z^2-d^2-2xy-2dz.$
23. $x^2-y^2+z^2-a^2-2xz+2ay.$
24. $2ab+a^2+b^2-c^2.$
25. $2xy-x^2-y^2+a^2+b^2-2ab.$
26. $(ax+by)^2-1.$

27. $1 - x^2 - y^2 + 2xy$.
28. $(5a - 2)^2 - (a - 4)^2$.
29. $a^2 - 2ab + b^2 - x^2$.
30. $(x + 1)^2 - (y + 1)^2$.
31. $(x + 1)^2 - (y - 1)^2$.
32. $d^2 - x^2 + 4xy - 4y^2$.
33. $a^2 - b^2 - 2bc - c^2$.
34. $4x^4 - 9x^2 + 6x - 1$.

133. CASE X.

Since $$\frac{x^3 - y^3}{x - y} = x^2 + xy + y^2,$$

and $$\frac{x^5 - y^5}{x - y} = x^4 + x^3y + x^2y^2 + xy^3 + y^4,$$

and so on, it follows that the difference between two equal odd powers of two numbers is divisible by the difference between the numbers.

EXERCISE XXXIX.

Resolve into factors:

1. $a^3 - b^3$.
2. $x^3 - 8$.
3. $x^3 - 343$.
4. $y^3 - 125$.
5. $y^3 - 216$.
6. $8x^3 - 27y^3$.
7. $64y^3 - 1000z^3$.
8. $729x^3 - 512y^3$.
9. $27a^3 - 1728$.
10. $1000a^3 - 1331b^3$.

134. CASE XI.

Since $$\frac{x^3 + a^3}{x + a} = x^2 - ax + a^2,$$

and $$\frac{x^5 + y^5}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4,$$

and so on, it follows that the sum of two equal odd powers of two numbers is divisible by the sum of the numbers.

EXERCISE XL.

Resolve into factors:

1. x^3+y^3.
2. x^3+8.
3. x^3+216.
4. y^3+64z^3.
5. $64b^3+125c^3$.
6. $216a^3+512c^3$.
7. $729x^3+1728y^3$.
8. x^5+y^5.
9. x^7+y^7.
10. $32b^5+243c^5$.

135. CASE XII. The sum of *any two powers* of two numbers, whose exponents contain the *same odd* factor, is divisible by the sum of the powers obtained by dividing the exponents of the given powers by this odd factor.

Thus,

$$\frac{x^6+y^6}{x^2+y^2}=x^4-x^2y^2+y^4.$$

$$\frac{x^6+y^9}{x^2+y^3}=x^4-x^2y^3+y^6.$$

In like manner, $x^{10}+32y^5$, which is equal to $x^{10}+(2y)^5$, is divisible by x^2+2y; but x^4+y^4, whose exponents do not contain *an odd* factor, and x^6+y^{10}, whose exponents do not contain the *same odd* factor, cannot be resolved into factors.

EXERCISE XLI.

Resolve into factors:

1. a^6+b^6.
2. $a^{10}+b^{10}$.
3. $x^{12}+y^{12}$.
4. b^6+64c^6.
5. x^6+1.
6. $a^{12}+1$.
7. $64a^6+x^6$.
8. $729+c^6$.

136. Case XIII. For a trinomial to be a perfect square, *the middle term must be twice the product of the square roots of the first and last terms.*

The expression $x^4 + x^2y^2 + y^4$ will become a perfect square if x^2y^2 be added to the middle term. And if the subtraction of x^2y^2 from the expression thus obtained be indicated, the result will be the difference of two squares. Thus:

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= (x^4 + 2x^2y^2 + y^4) - x^2y^2, \\ &= (x^2 + y^2)^2 - x^2y^2, \\ &= (x^2 + y^2 + xy)(x^2 + y^2 - xy), \\ \text{or, } &(x^2 + xy + y^2)(x^2 - xy + y^2). \end{aligned}$$

Exercise XLII.

Resolve into factors:

1. $a^4 + a^2b^2 + b^4$.
2. $9x^4 + 3x^2y^2 + 4y^4$.
3. $16x^4 - 17x^2y^2 + y^4$.
4. $81a^4 + 23a^2b^2 + 16b^4$.
5. $81a^4 - 28a^2b^2 + 16b^4$.
6. $9x^4 + 38x^2y^2 + 49y^4$.
7. $25a^4 - 9a^2b^2 + 16b^4$.
8. $49m^4 + 110m^2n^2 + 81n^4$.
9. $9a^4 + 21a^2c^2 + 25c^4$.
10. $49a^4 - 15a^2b^2 + 121b^4$.
11. $64x^4 + 128x^2y^2 + 81y^4$.
12. *$4x^4 - 37x^2y^2 + 9y^4$.
13. $25x^4 - 41x^2y^2 + 16y^4$.
14. $81x^4 - 34x^2y^2 + y^4$.

* If, in Example 12, $9y^4 = (-3y^2)^2$, then $25x^2y^2$ should be added to $4x^4 - 37x^2y^2 + 9y^4$, in order to make the expression a perfect square. That is, we should have:

$$\begin{aligned} &(4x^4 - 12x^2y^2 + 9y^4) - 25x^2y^2, \\ &= (2x^2 - 3y^2)^2 - 25x^2y^2, \\ &= (2x^2 - 3y^2 + 5xy)(2x^2 - 3y^2 - 5xy), \\ \text{or, } &(2x^2 + 5xy - 3y^2)(2x^2 - 5xy - 3y^2). \end{aligned}$$

137. CASE XIV. To find the factors of

$$6x^2 + x - 12.$$

It is evident that the first terms of the two factors might be $6x$ and x, or $2x$ and $3x$, since the product of either of these pairs is $6x^2$.

Likewise, the last terms of the two factors might be 12 and 1, 6 and 2, or 4 and 3 (if we disregard the signs).

From these it is necessary to select such as will produce the middle term of the trinomial. And they are found by trial to be $3x$ and $2x$, and -4 and $+3$.

$$\therefore 6x^2 + x - 12 = (3x - 4)(2x + 3).$$

EXERCISE XLIII.

Resolve into factors:

1. $12x^2 - 5x - 2.$
2. $12x^2 - 7x + 1.$
3. $12x^2 - x - 1.$
4. $3x^2 - 2x - 5.$
5. $3x^2 + 4x - 4.$
6. $6x^2 + 5x - 4.$
7. $4x^2 + 13x + 3.$
8. $4x^2 + 11x - 3.$
9. $4x^2 - 4x - 3.$
10. $x^2 - 3ax + 2a^2.$
11. $12a^4 + a^2x^2 - x^4.$
12. $2x^2 + 5xy + 2y^2.$
13. $6a^2x^2 + ax - 1.$
14. $6b^2 - 7bx - 3x^2.$
15. $4x^2 + 8x + 3.$
16. $a^2 - ax - 6x^2.$
17. $8a^2 + 14ab - 15b^2.$
18. $6a^2 - 19ac + 10c^2.$
19. $8x^2 + 34xy + 21y^2.$
20. $8x^2 - 22xy - 21y^2.$
21. $6x^2 + 19xy - 7y^2.$
22. $11a^2 - 23ab + 2b^2.$
23. $2c^2 - 13cd + 6d^2.$
24. $6y^2 + 7yz - 3z^2.$

138. CASE XV. The factors, if any exist, of a polynomial of more than three terms can often be found by the application of principles already explained. Thus it is seen at a glance that the expression

$$a^3 - 3a^2b + 3ab^2 - b^3$$

fulfils, both in respect to exponents and coefficients, the laws stated in § 83 for writing the power of a binomial; and it is known at once that

$$a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3.$$

Again, it is seen that the expression

$$x^2 - 2xy + y^2 + 2xz - 2yz + z^2$$

consists of *three* squares and *three* double products, and from § 79, is the square of a *trinomial* which has for terms x, y, z.

It is also seen from the double product $-2xy$, that x and y have *unlike* signs;

and from the double product $2xz$, that x and z have *like* signs. Hence,

$$x^2 - 2xy + y^2 + 2xz - 2yz + z^2 = (x - y + z)^2.$$

EXERCISE XLIV.

Resolve into factors:

1. $a^3 + 3a^2b + 3ab^2 + b^3$.
2. $a^3 + 3a^2 + 3a + 1$.
3. $a^3 - 3a^2 + 3a - 1$.
4. $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.
5. $x^4 - 4x^3 + 6x^2 - 4x + 1$.
6. $a^4 - 4a^3c + 6a^2c^2 - 4ac^3 + c^4$.
7. $x^2 + 2xy + y^2 + 2xz + 2yz + z^2$.
8. $x^2 - 2xy + y^2 - 2xz + 2yz + z^2$.
9. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.

139. CASE XVI. Multiply $2x - y + 3$ by $x + 2y - 3$.

$$\begin{array}{llllll}
2x & -\ y & +3 & & & \\
x & +\ 2y & -3 & & & \\
\hline
2x^2 & -\ xy & & +3x & & \\
 & \ \ 4xy & -2y^2 & & +6y & \\
 & & & -6x & +3y & -9 \\
\hline
2x^2 & +3xy & -2y^2 & -3x & +9y & -9
\end{array}$$

It is to be observed that $2x^2+3xy-2y^2$, of the product, is obtained from $(2x-y)\times(x+2y)$;

that -9 is obtained from 3×-3;

that $-3x$ is the sum of $2x\times-3$ and $x\times3$;

that $9y$ is the sum of $2y\times3$ and $-y\times-3$.

From this result may be deduced a method of resolving into its factors a polynomial which is composed of two trinomial factors. Thus:

Find the factors of

$$6x^2 - 7xy - 3y^2 - 9x + 30y - 27.$$

The factors of the first three terms are (by Case XIV.)

$$3x + y \text{ and } 2x - 3y.$$

Now -27 must be resolved into two factors such that the sum of the products obtained by multiplying one of these factors by $3x$ and the other by $2x$ shall be $-9x$.

These two factors evidently are -9 and $+3$.

That is, $(6x^2 - 7xy - 3y^2 - 9x + 30y - 27)$
$$= (3x + y - 9)(2x - 3y + 3).$$

140. The following method is often most convenient for separating a polynomial into its factors:

Find the factors of

$$2x^2 - 5xy + 2y^2 + 7xz - 5yz + 3z^2.$$

1. Reject the terms that contain z.
2. Reject the terms that contain y.
3. Reject the terms that contain x.

Factor the expression that remains in each case.

1. $2x^2 - 5xy + 2y^2 = (x - 2y)(2x - y)$.
2. $2x^2 + 7xz + 3z^2 = (x + 3z)(2x + z)$.
3. $2y^2 - 5yz + 3z^2 = (-2y + 3z)(-y + z)$.

Arrange these three pairs of factors in two rows of three factors each, so that any two factors of each row may have a *common term*. Thus:

$$x - 2y,\ x + 3z,\ -2y + 3z;$$
$$2x - y,\ 2x + z,\ -y + z.$$

From the first row, select the *terms common to two factors* for one trinomial factor:

$$x - 2y + 3z.$$

From the second row, select the *terms common to two factors* for the other trinomial factor:

$$2x - y + z.$$

Then, $2x^2 - 5xy + 2y^2 + 7xz - 5yz + 3z^2$
$$= (x - 2y + 3z)(2x - y + z).$$

141. When a factor obtained from the first three terms is also a factor of the remaining terms, the expression is easily resolved. Thus:

$$\begin{aligned}(3)\ & x^2 - 3xy + 2y^2 - 3x + 6y,\\ &= (x - 2y)(x - y) - 3(x - 2y),\\ &= (x - 2y)(x - y - 3).\end{aligned}$$

Exercise XLV.

Resolve into factors:

1. $2x^2 - 5xy + 2y^2 - 17x + 13y + 21.$
2. $6x^2 - 37xy + 6y^2 - 5x - 5y - 1.$
3. $6x^2 - 5xy - 6y^2 - x - 5y - 1.$
4. $5x^2 - 8xy + 3y^2 + 7x - 5y + 2.$
5. $2x^2 - xy - 3y^2 - 8x + 7y + 6.$

6. $x^2 - 25y^2 - 10x - 20y + 21.$

7. $2x^2 - 5xy + 2y^2 - xz - yz - z^2.$

8. $6x^2 + xy - y^2 - 3xz + 6yz - 9z^2.$

9. $6x^2 - 7xy + y^2 + 35xz - 5yz - 6z^2.$

10. $5x^2 - 8xy + 3y^2 - 3xz + yz - 2z^2.$

11. $2x^2 - xy - 3y^2 - 5yz - 2z^2.$

12. $6x^2 - 13xy + 6y^2 + 12xz - 13yz + 6z^2.$

13. $x^2 - 2xy + y^2 + 5x - 5y.$

14. $2x^2 + 5xy - 3y^2 - 4xz + 2yz.$

Exercise XLVI.

MISCELLANEOUS EXAMPLES.

The following expressions are to be resolved into factors by the principles already explained. The student should first carefully remove all monomial factors from the expressions.

1. $5x^2 - 15x - 20.$
2. $2x^5 - 16x^4 + 24x^3.$
3. $3a^2b^2 - 9ab - 12.$
4. $a^2 + 2ax + x^2 + 4a + 4x.$
5. $a^2 - 2ab + b^2 - c^2.$
6. $x^2 - 2xy + y^2 - c^2 + 2cd - d^2.$
7. $4 - x^2 - 2x^3 - x^4.$
8. $a^2 - b^2 - a - b.$
9. $a^4 + a^2 + 1.$
10. $x^2 - y^2 - xz + yz.$
11. $ab - ac - b^2 + bc.$
12. $3x^2 - 3xz - xy + yz.$
13. $a^2 - x^2 - ab - bx.$
14. $a^2 - 2ax + x^2 + a - x.$
15. $3x^2 - 3y^2 - 2x + 2y.$
16. $x^4 + x^3 + x^2 + x.$
17. $a^4x^4 - a^3x^3 - a^2x^2 + 1.$
18. $3x^3 - 2x^2y - 27xy^2 + 18y^3.$

19. $4x^4-x^2+2x-1.$

20. $x^6-y^6.$

21. $x^6+y^6.$

22. $729-x^6.$

23. $x^{12}y+y^{13}.$

24. $a^4c-c^5.$

25. $x^2+4x-21.$

26. $3a^2-21ab+30b^2.$

27. $2x^4-4x^3y-6x^2y^2.$

28. $4a^2-4ab+b^2.$

29. $16x^2-80xy+100y^2.$

30. $36a^2x^3y^2-25b^2x^5y^2.$

31. $9x^2y^4-30xy^2z+25z^2.$

32. $16x^5-x.$

33. $x^2-2xy-2xz+y^2+2yz+z^2.$

34. $a^2-ab-6b^2-4a+12b.$

35. $x^2+2xy+y^2-x-y-6.$

36. $(a+b)^4-c^4.$

37. $x^2-xy-6y^2-4x+12y.$

38. $1-x+x^2-x^3.$

39. $3x^2-11xy+6y^2.$

40. $x^2+20x+91.$

41. $(x-y)(x^2-z^2)-(x-z)(x^2-y^2).$

42. $x^2-5x-24.$

43. $(x^2-y^2-z^2)^2-4y^2z^2.$

44. $5x^3y^2+5x^2yz-60xz^2.$

45. $3x^3-x^2+3x-1.$

46. $x^2-2mx+m^2-n^2.$

47. $4a^2b^2-(a^2+b^2-c^2)^2.$

48. $a^7+a^5.$

49. $1-14a^3x+49a^6x^2.$

50. $y^2-4y-117.$

51. $x^2+6x-135.$

52. $4a^2-12ab+9b^2-4c^2.$

53. $(a+3b)^2-9(b-c)^2.$

54. $9x^2-4y^2+4yz-z^2.$

55. $6b^2x^2-7bx^3-3x^4.$

56. $a^3-b^3-3ab(a-b).$

57. $x^3+y^3+3xy(x+y).$

58. $a^3-b^3-a(a^2-b^2)+b(a-b)^2.$

59. $9x^2y^2-3xy^3-6y^4.$

60. $6x^2+13xy+6y^2.$

61. $6a^2b^2-ab^3-12b^4.$

62. $a^2+2ad+d^2-4b^2+12bc-9c^2.$

63. $x^3-2x^2y+4xy^2-8y^3.$

64. $4a^2x^2-8abx+3b^2.$

65. $18x^2-24xy+8y^2+9x-6y$.

66. $2x^2+2xy-12y^2+6xz+18yz$.

67. $(x+y)^2-1-xy(x+y+1)$.

68. $x^2-y^2-z^2+2yz+x+y-z$.

69. $2x^2+4xy+2y^2+2ax+2ay$.

70. $16a^2b+32abc+12bc^2$.

71. $m^2p-m^2q-n^2p+n^2q$.

72. $12ax^2-14axy-6ay^2$.

73. $2x^3+4x^2-70x$.

74. $16a^3x-2x^4$.

75. $32bx^3-4by^3$.

76. $x-27x^4$.

77. $x^{12}-y^{12}$.

78. $49m^2-121n^2$.

79. $16-81y^4$.

80. $12z^4-z^2-6$.

81. x^3-x^2+x-1.

82. $x^2+2x+1-y^2$.

83. $49(a-b)^2-64(m-n)^2$.

84. $1-\left(\dfrac{a^2+b^2-c^2}{2ab}\right)^2$.

85. $x^2-53x+360$.

86. $x^3-2x^2y+x^2-4x+8y-4$.

87. $2ab-2bc-ae+ce+2b^2-be$.

88. $125x^5+350x^3y^2+245xy^4$.

89. $a^6+a^5b+a^4b^2+a^3b^3+a^2b^4+ab^5$.

90. $2a^4x-2a^3cx+2ac^3x-2c^4x$.

91. $6x^2-5xy-6y^2+3xz+15yz-9z^2$.

92. $4x^2-9xy+2y^2-3xz-yz-z^2$.

93. $3a^2-7ab+2b^2+5ac-5bc+2c^2$.

94. $x^4-2x^3+x^2-8x+8$.

95. $5x^2-8xy+3y^2-5x+3y$.

96. $a^2-2ad+d^2-4b^2+12bc-9c^2$.

97. $(x^2-x-6)(x^2-x-20)$.

CHAPTER VII.

Common Factors and Multiples.

142. A **common factor** of two or more expressions is an expression which is contained in each of them without a remainder. Thus,

$5a$ is a common factor of $20a$ and $25a$;

$3x^2y^2$ is a common factor of $12x^2y^3$ and $15x^3y^3$.

143. Two expressions which have no common factor except 1, are said to be *prime* to each other.

144. The **Highest Common Factor** of two or more expressions is the product of all the factors common to the expressions.

Thus, $3a^2$ is the highest common factor of $3a^2$, $6a^3$, and $12a^4$.

$5x^2y^2$ is the highest common factor of $10x^3y^2$ and $15x^2y^3$.

For brevity, H. C. F. will be used for Highest Common Factor.

(1) Find the H. C. F. of $42a^3b^2x$ and $21a^2b^3x^2$.

$$42a^3b^2x = 2 \times 3 \times 7 \times a^3 \times b^2 \times x;$$
$$21a^2b^3x^2 = 3 \times 7 \times a^2 \times b^3 \times x^2.$$
$$\therefore \text{ the H. C. F.} = 3 \times 7 \times a^2 \times b^2 \times x.$$
$$= 21a^2b^2x.$$

(2) Find the H. C. F. of $2a^2x + 2ax^2$ and $3abxy + 3bx^2y$.

$$2a^2x + 2ax^2 = 2ax(a + x);$$
$$3abxy + 3bx^2y = 3bxy(a + x).$$
$$\therefore \text{ the H. C. F.} = x(a + x).$$

(3) Find the H. C. F. of

$$8a^2x^2 - 24a^2x + 16a^2 \text{ and } 12ax^2y - 12axy - 24ay.$$

$$\begin{aligned} 8a^2x^2 - 24a^2x + 16a^2 &= 8a^2(x^2 - 3x + 2), \\ &= 2^3a^2(x-1)(x-2); \\ 12ax^2y - 12axy - 24ay &= 12ay(x^2 - x - 2), \\ &= 2^2 \times 3ay(x+1)(x-2). \\ \therefore \text{ the H. C. F.} &= 2^2a(x-2), \\ &= 4a(x-2). \end{aligned}$$

Hence, to find the H. C. F. of two or more expressions:

Resolve each expression into its lowest factors.

Select from these the lowest power of each common factor, and find the product of these powers.

Exercise XLVII.

Find the H. C. F. of:

1. $18ab^2c^3d$ and $36a^2bcd^2$.
2. $17pq^2$, $34p^2q$, and $51p^3q^3$.
3. $8x^2y^3z^4$, $12x^3y^2z^3$, and $20x^4y^3z^2$.
4. $30x^4y^5$, $90x^2y^3$, and $120x^3y^4$.
5. $a^2 - b^2$ and $a^3 - b^3$.
6. $a^2 - x^2$ and $(a-x)^2$.
7. $a^3 + x^3$ and $(a+x)^3$.
8. $9x^2 - 1$ and $(3x+1)^2$.
9. $7x^2 - 4x$ and $7a^2x - 4a^2$.
10. $12a^3x^2y - 4a^3xy^2$ and $30a^2x^3y^2 - 10a^2x^2y^3$.
11. $8a^3b^2c - 12a^2bc^3$ and $6ab^4c + 4ab^3c^2$.
12. $x^2 - 2x - 3$ and $x^2 + x - 12$.
13. $2a^3 - 2ab^2$ and $4b(a+b)^2$.
14. $12x^3y(x-y)(x-3y)$ and $18x^2(x-y)(3x-y)$.
15. $3x^3 + 6x^2 - 24x$ and $6x^3 - 96x$.

16. $ac(a-b)(a-c)$ and $bc(b-a)(b-c)$.

17. $10x^3y-60x^2y^2+5xy^3$ and $5x^2y^2-5xy^3-100y^4$.

18. $x(x+1)^2$, $x^2(x^2-1)$, and $2x(x^2-x-2)$.

19. $3x^2-6x+3$, $6x^2+6x-12$, and $12x^2-12$.

20. $6(a-b)^4$, $8(a^2-b^2)^2$, and $10(a^4-b^4)$.

21. x^2-y^2, $(x+y)^2$, and $x^2+3xy+2y^2$.

22. x^2-y^2, x^3-y^3, and $x^2-7xy+6y^2$.

23. x^2-1, x^3-1, and x^2+x-2.

145. When it is required to find the H. C. F. of two or more expressions which cannot readily be resolved into their factors, the method to be employed is similar to that of the corresponding case in arithmetic. And as that method consists in obtaining pairs of continually decreasing numbers which contain as a factor the H. C. F. requir d; so in algebra, pairs of expressions of continually decreasing degrees are obtained, which contain as a factor the H. C. F. required.

The method depends upon two principles:

1. *Any factor of an expression is a factor also of any multiple of that expression.*

Thus, if F represent a factor of an expression A, so that $A=nF$, then $mA=mnF$. That is, mA contains the factor F.

2. *Any common factor of two expressions is a factor of the sum or difference of any multiples of the expressions.*

Thus, if F represent a common factor of the expressions A and B so that

$$A=mF, \text{ and } B=nF;$$

then $\qquad pA=pmF, \text{ and } qB=qnF.$

Hence, $\qquad pA\pm qB=pmF\pm qnF,$

$$=(pm\pm qn)F.$$

That is, $\qquad pA\pm qB$ contains the factor F.

146. The general proof of this method as applied to *numbers* is as follows:

Let a and b be two numbers, of which a is the greater. The operation may be represented by:

$$\begin{array}{lll}
b)\,a\,(p & \quad 42)\,154\,(3 & \quad nF)\,mF\,(p \\
\ \ \underline{pb} & \quad \quad \underline{126} & \quad \quad \underline{pnF} \\
\ \ \ c)\,b\,(q & \quad \quad 28)\,42\,(1 & \quad \quad cF)\,nF\,(q \\
\ \ \ \ \ \underline{qc} & \quad \quad \quad \underline{28} & \quad \quad \quad \underline{qcF} \\
\ \ \ \ \ d)\,c\,(r & \quad \quad \quad 14)\,28\,(2 & \quad \quad \quad F)\,cF\,(c \\
\ \ \ \ \ \ \ \underline{rd} & \quad \quad \quad \quad \underline{28} & \quad \quad \quad \quad \underline{cF}
\end{array}$$

p, q, and r represent the several quotients,
c and d represent the remainders,
and d is supposed to be contained exactly in c.

The numbers represented are all integral.

Then
$$\begin{aligned}
c &= rd, \\
b &= qc + d = qrd + d = (qr+1)d, \\
a &= pb + c = pqrd + pd + rd, \\
&= (pqr + p + r)d.
\end{aligned}$$

$\therefore$ d is *a common factor* of a and b.

It remains to show that d is the *highest* common factor of a and b.

Let f represent the highest common factor of a and b.

Now $c = a - pb$, and f is a common factor of a and b.

$\therefore$ by (2) f is a factor of c.

Also, $d = b - qc$, and f is a common factor of b and c.

$\therefore$ by (2) f is a factor of d.

That is, d *contains the highest common factor* of a and b.

But it has been shown that d is a common factor of a and b.

$\therefore$ d is the *highest common factor* of a and b.

NOTE. The second operation represents the application of the method to a particular case. The third operation is intended to represent clearly that every remainder in the course of the operation contains as a factor the H. C. F. sought, and that this is the *highest factor common* to that remainder and the preceding divisor.

147. By the same method, find the H. C. F. of

$$2x^2+x-3 \text{ and } 4x^3+8x^2-x-6.$$

$$\begin{array}{rl}
2x^2+x-3) & 4x^3+8x^2-x-6(2x+3 \\
 & \underline{4x^3+2x^2-6x} \\
 & \quad 6x^2+5x-6 \\
 & \quad \underline{6x^2+3x-9} \\
 & \qquad 2x+3)2x^2+x-3(x-1 \\
 & \qquad\qquad \underline{2x^2+3x} \\
 & \qquad\qquad -2x-3 \\
 & \qquad\qquad \underline{-2x-3}
\end{array}$$

$\therefore$ the H. C. F. $=2x+3.$

The given expressions are arranged according to the descending powers of x.

The expression whose first term is of the lower degree is taken for the divisor; and each division is continued until the first term of the remainder is of lower degree than that of the divisor.

148. This method is of use only to determine the **compound** factor of the H. C. F. **Simple** factors of the given expressions must first be separated from them, and the highest common factor of these must be reserved to be multiplied into the compound factor obtained.

Find the H. C. F. of

$$12x^4+30x^3-72x^2 \text{ and } 32x^3+84x^2-176x.$$

$12x^4+30x^3-72x^2=6x^2(2x^2+5x-12).$

$32x^3+84x^2-176x=4x(8x^2+21x-44).$

$6x^2$ and $4x$ have $2x$ *common.*

$$\begin{array}{rl}
2x^2+5x-12) & 8x^2+21x-44(4 \\
 & \underline{8x^2+20x-48} \\
 & \qquad x+4)2x^2+5x-12(2x-3 \\
 & \qquad\qquad \underline{2x^2+8x} \\
 & \qquad\qquad -3x-12 \\
 & \qquad\qquad \underline{-3x-12}
\end{array}$$

$\therefore$ the H. C. F. $=2x(x+4).$

149. Modifications of this method are sometimes needed.

(1) Find the H. C. F. of $4x^2-8x-5$ and $12x^2-4x-65$.

$$
\begin{array}{r|l}
4x^2-8x-5) & 12x^2-\ \ 4x-65\,(3 \\
& \underline{12x^2-24x-15} \\
& \qquad\ \ 20x-50
\end{array}
$$

The first division ends here, for $20x$ is of lower degree than $4x^2$. But if $20x-50$ be made the divisor, $4x^2$ will not contain $20x$ an *integral* number of times.

Now, it is to be remembered that the H. C. F. sought *is contained in the remainder* $20x-50$, and that it is a *compound factor*. Hence if the *simple factor* 10 be removed, the H. C. F. must still be contained in $2x-5$, and therefore the process may be continued with $2x-5$ for a divisor.

$$
\begin{array}{r|l}
2x-5) & 4x^2-\ \ 8x-5\,(2x+1 \\
& \underline{4x^2-10x\qquad} \\
& \qquad\ \ 2x-5 \\
& \qquad\ \ \underline{2x-5}
\end{array}
$$

$$\therefore \text{ the H. C. F.} = 2x-5.$$

(2) Find the H. C. F. of

$$21x^3-4x^2-15x-2 \text{ and } 21x^3-32x^2-54x-7.$$

$$
\begin{array}{r|l}
21x^3-4x^2-15x-2) & 21x^3-32x^2-54x-7\,(1 \\
& \underline{21x^3-\ \ 4x^2-15x-2} \\
& \qquad\ -28x^2-39x-5
\end{array}
$$

The difficulty here cannot be obviated by *removing* a simple factor from the remainder, for $-28x^2-39x-5$ has no simple factor. In this case, the expression $21x^3-4x^2-15x-2$ must be *multiplied* by the simple factor 4 to make its first term divisible by $-28x^2$.

The *introduction* of such a factor can in no way affect the H. C. F. sought; for the H. C. F. contains only factors *common to the remainder and the last divisor*, and 4 is not a factor of the remainder.

The *signs* of all the terms of the remainder may be changed; for if an expression A is divisible by $-F$, it is divisible by $+F$.

The process then is continued by changing the signs of the remainder and multiplying the divisor by 4.

$$
\begin{array}{lr}
 & 28x^2+39x+5\,)\,84x^3-\ \ 16x^2-\ \ 60x-\ \ 8\,(\,3x \\
 & \underline{84x^3+117x^2+\ \ 15x\qquad} \\
 & -133x^2-\ \ 75x-\ \ 8 \\
\text{Multiply by } -4, & \underline{-4\qquad\qquad\qquad} \\
 & 532x^2+300x+32\,(\,19 \\
 & \underline{532x^2+741x+95} \\
\text{Divide by } -63, & -63\,)\,\underline{-441x-63} \\
 & 7x+\ \ 1
\end{array}
$$

$$
\begin{array}{lr}
 & 7x+1)\,28x^2+39x+5\,(\,4x+5 \\
 & \underline{28x^2+\ \ 4x\qquad} \\
 & 35x+5 \\
\therefore \text{ the H. C. F.} = 7x+1. & \underline{35x+5}
\end{array}
$$

(3) Find the H. C. F. of

$$8x^2+2x-3 \text{ and } 6x^3+5x^2-2.$$

$$
\begin{array}{lr}
 & 6x^3+\ \ 5x^2-\ \ 2\qquad\qquad \\
 & 4\qquad\qquad\qquad\qquad \\
 & 8x^2+2x-3\,)\,24x^3+20x^2-\ \ 8\ \,(\,3x+7 \\
 & \underline{24x^3+\ \ 6x^2-\ \ 9x}\qquad \\
 & 14x^2+\ \ 9x-\ \ 8\qquad \\
\text{Multiply by } 4, & \underline{4\qquad\qquad\qquad}\qquad \\
 & 56x^2+36x-32\qquad \\
 & \underline{56x^2+14x-21}\qquad \\
\text{Divide by } 11, & 11\,)\,\underline{22x-11}\qquad \\
 & 2x-\ \ 1\,)\,8x^2+2x-3\,(\,4x+3 \\
 & \underline{8x^2-4x\qquad} \\
 & 6x-3 \\
\therefore \text{ the H. C. F.} = 2x-1. & \underline{6x-3}
\end{array}
$$

In this case it is necessary to multiply by 4 the *given expression* $6x^3+5x^2-2$ to make its first term divisible by $8x^2$, 4 being obviously not a *common* factor.

The following arrangement of the work will be found most convenient:

$$
\begin{array}{r|r|l}
8x^2+2x-3 & 6x^3+\ \ 5x^2-\ \ 2 & \\
\underline{8x^2-4x} & \underline{4} & \\
6x-3 & 24x^3+20x^2-\ \ 8 & 3x \\
\underline{6x-3} & \underline{24x^3+\ \ 6x^2-\ \ 9x} & \\
 & 14x^2+\ \ 9x-\ \ 8 & \\
 & \underline{4} & \\
 & 56x^2+36x-32 & +7 \\
 & \underline{56x^2+14x-21} & \\
 & 11\overline{)22x-11} & \\
 & 2x-\ \ 1 & 4x+3
\end{array}
$$

150. From the foregoing examples it will be seen that, in the algebraic process of finding the highest common factor, the following steps, in the order here given, must be carefully observed:

I. Simple factors of the given expressions are to be removed from them, and the highest common factor of these is to be reserved as a factor of the H. C. F. sought.

II. The resulting compound expressions are to be arranged according to the *descending* powers of a common letter; and that expression which is of the lower degree is to be taken for the divisor; or, if both are of the same degree, that whose first term has the smaller coefficient.

III. Each division is to be continued until the remainder is of lower degree than the divisor.

IV. If the final remainder of any division is found to contain a factor that is not a *common* factor of the given expressions, *this factor is to be removed;* and the resulting expression is to be used as the next divisor.

V. A dividend whose first term is not exactly divisible by the first term of the divisor, is to be *multiplied* by such an expression as will make it thus divisible.

Exercise XLVIII.

Find the H. C. F. of:

1. $5x^2+4x-1, \quad 20x^2+21x-5.$
2. $2x^3-4x^2-13x-7, \quad 6x^3-11x^2-37x-20.$
3. $6a^4+25a^3-21a^2+4a, \quad 24a^4+112a^3-94a^2+18a.$
4. $9x^3+9x^2-4x-4, \quad 45x^3+54x^2-20x-24.$
5. $27x^5-3x^4+6x^3-3x^2, \quad 162x^6+48x^3-18x^2+6x.$
6. $20x^3-60x^2+50x-20, \quad 32x^4-92x^3+68x^2-24x.$
7. $4x^2-8x-5, \quad 12x^2-4x-65.$
8. $3a^3-5a^2x-2ax^2, \quad 9a^3-8a^2x-20ax^2.$
9. $10x^3+x^2-9x+24, \quad 20x^4-17x^2+48x-3.$
10. $8x^3-4x^2-32x-182, \quad 36x^3-84x^2-111x-126.$
11. $5x^2(12x^3+4x^2+17x-3), \quad 10x(24x^3-52x^2+14x-1).$
12. $9x^4y-x^2y^3-20xy^4, \quad 18x^3y-18x^2y^2-2xy^3-8y^4.$
13. $6x^2-x-15, \quad 9x^2-3x-20.$
14. $12x^3-9x^2+5x+2, \quad 24x^2+10x+1.$
15. $6x^3+15x^2-6x+9, \quad 9x^3+6x^2-51x+36.$
16. $4x^3-x^2y-xy^2-5y^3, \quad 7x^3+4x^2y+4xy^2-3y^3.$
17. $2a^3-2a^2-3a-2, \quad 3a^3-a^2-2a-16.$
18. $12y^3+2y^2-94y-60, \quad 48y^3-24y^2-348y+30.$
19. $9x(2x^4-6x^3-x^2+15x-10),$
    $6x^2(4x^4+6x^3-4x^2-15x-15).$
20. $15x^4+2x^3-75x^2+5x+2, \quad 35x^4+x^3-175x^2+30x+1.$
21. $21x^4-4x^3-15x^2-2x, \quad 21x^3-32x^2-54x-7.$
22. $9x^4y-22x^2y^3-3xy^4+10y^5, \quad 9x^5y-6x^4y^2+x^3y^3-25xy^5.$

23. $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1,$
$4x^4 + 2x^3 - 18x^2 + 3x - 5.$

24. $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4,\ \ 3x^3 - 7ax^2 + 3a^2x - 2a^3.$

151. The H. C. F. of three expressions will be obtained by finding the H. C. F. of two of them, and then of that and the third expression.

For, if A, B, and C are three expressions,
and D the highest common factor of A and B,
and E the highest common factor of D and C,
Then D contains every factor common to A and B,
and E contains every factor common to D and C.
$\therefore$ E contains every factor common to A, B, and C.

Exercise XLIX.

Find the H. C. F. of:

1. $2x^2 + x - 1,\ \ x^2 + 5x + 4,\ \ x^3 + 1.$
2. $y^3 - y^2 - y + 1,\ \ 3y^2 - 2y - 1,\ \ y^3 - y^2 + y - 1.$
3. $x^3 - 4x^2 + 9x - 10,\ \ x^3 + 2x^2 - 3x + 20,\ \ x^3 + 5x^2 - 9x + 35.$
4. $x^3 - 7x^2 + 16x - 12,\ \ 3x^3 - 14x^2 + 16x,$
$5x^3 - 10x^2 + 7x - 14.$
5. $y^3 - 5y^2 + 11y - 15,\ \ y^3 - y^2 + 3y + 5,$
$2y^3 - 7y^2 + 16y - 15.$
6. $2x^2 + 3x - 5,\ \ 3x^2 - x - 2,\ \ 2x^2 + x - 3.$
7. $x^3 - 1,\ \ x^3 - x^2 - x - 2,\ \ 2x^3 - x^2 - x - 3.$
8. $x^3 - 3x - 2,\ \ 2x^3 + 3x^2 - 1,\ \ x^3 + 1.$
9. $12(x^4 - y^4),\ \ 10(x^6 - y^6),\ \ 8(x^4y + xy^4).$
10. $x^4 + xy^3,\ \ x^3y + y^4,\ \ x^4 + x^2y^2 + y^4.$
11. $2(x^2y - xy^2),\ \ 3(x^3y - xy^3),\ \ 4(x^4y - xy^4),\ \ 5(x^5y - xy^5).$

Lowest Common Multiple.

152. A **common multiple** of two or more expressions is an expression which is exactly divisible by each of them.

153. The **Lowest Common Multiple** of two or more expressions is the product of all the factors of the expressions, each factor being written with its highest exponent.

154. The lowest common multiple of two expressions which have no common factor will be their product.

For brevity L. C. M. will be used for Lowest Common Multiple.

(1) Find the L. C. M. of $12a^2c$, $14bc^2$, $36ab^2$.

$$\begin{aligned} 12a^2c &= 2^2 \times 3a^2c, \\ 14bc^2 &= 2 \times 7bc^2, \\ 36ab^2 &= 2^2 \times 3^2ab^2. \end{aligned}$$

$\therefore$ the L. C. M. $= 2^2 \times 3^2 \times 7a^2b^2c^2 = 252a^2b^2c^2$.

(2) Find the L. C. M. of

$$2a^2+2ax,\ 6a^2-6x^2,\ 3a^2-6ax+3x^2.$$

$$\begin{aligned} 2a^2+2ax &= 2a(a+x), \\ 6a^2-6x^2 &= 2 \times 3(a+x)(a-x), \\ 3a^2-6ax+3x^2 &= 3(a-x)^2. \end{aligned}$$

$\therefore$ the L. C. M. $= 6a(a+x)(a-x)^2$.

Exercise L.

Find the L. C. M. of:

1. $4a^3x$, $6a^2x^2$, $2ax^2$.
2. $18ax^2$, $72ay^2$, $12xy$.
3. x^2, $ax+x^2$.
4. x^2-1, x^2-x.
5. a^2-b^2, a^2+ab.
6. $2x-1$, $4x^2-1$.

7. $a+b,\ a^3+b^3.$

8. $x^2-1,\ x^2+1,\ x^4-1.$

9. $x^2-x,\ x^3-1,\ x^3+1.$

10. $x^2-1,\ x^2-x,\ x^3-1.$

11. $2a+1,\ 4a^2-1,\ 8a^3+1.$

12. $(a+b)^2,\ a^2-b^2.$

13. $4(1+x),\ 4(1-x),\ 2(1-x^2).$

14. $x-1,\ x^2+x+1,\ x^3-1.$

15. $x^2-y^2,\ (x+y)^2,\ (x-y)^2.$

16. $x^2-y^2,\ 3(x-y)^2,\ 12(x^3+y^3).$

17. $6(x^2+xy),\ 8(xy-y^2),\ 10(x^2-y^2).$

18. $x^2+5x+6,\ x^2+6x+8.$

19. $a^2-a-20,\ a^2+a-12.$

20. $x^2+11x+30,\ x^2+12x+35.$

21. $x^2-9x-22,\ x^2-13x+22.$

22. $4ab(a^2-3ab+2b^2),\ 5a^3(a^2+ab-6b^2).$

23. $20(x^2-1),\ 24(x^2-x-2),\ 16(x^2+x-2).$

24. $12xy(x^2-y^2),\ 2x^2(x+y)^2,\ 3y^2(x-y)^2.$

25. $(a-b)(b-c),\ (b-c)(c-a),\ (c-a)(a-b).$

26. $(a-b)(a-c),\ (b-a)(b-c),\ (c-a)(c-b).$

27. $x^3-4x^2+3x,\ x^4+x^3-12x^2,\ x^5+3x^4-4x^3.$

28. $x^2y-xy^2,\ 3x(x-y)^2,\ 4y(x-y)^3.$

29. $(a+b)^2-(c+d)^2,\ (a+c)^2-(b+d)^2,\ (a+d)^2-(b+c)^2.$

30. $(2x-4)(3x-6),\ (x-3)(4x-8),\ (2x-6)(5x-10).$

155. When the expressions cannot be readily resolved into their factors, the expressions may be resolved by finding their H. C. F.

I. Find the L. C. M. of

$$6x^3-11x^2y+2y^3 \text{ and } 9x^3-22xy^2-8y^3.$$

$$\begin{array}{r|l|l}
6x^3-11x^2y \qquad\quad +2y^3 & 9x^3-22xy^2-8y^3 & 3 \\
6x^3-8x^2y-4xy^2 \qquad & 2 & \\
\hline
-\;3x^2y+4xy^2+2y^3 & 18x^3-44xy^2-16y^3 & \\
-\;3x^2y+4xy^2+2y^3 & 18x^3-33x^2y+6y^3 & \\
\hline
 & 11y\,)\,33x^2y-44xy^2-22y^3 & \\
 & \qquad\quad 3x^2\;-\;4xy\;-\;2y^2 & 2x-y
\end{array}$$

Hence, $6x^3-11x^2y+2y^3=(2x-y)\;(3x^2-4xy-2y^2)$,
and $9x^3-22xy^2-8y^3=(3x+4y)(3x^2-4xy-2y^2)$.

$\therefore$ the L. C. M. $=(2x-y)(3x+4y)(3x^2-4xy-2y^2)$.

In this example we find the H. C. F. of the given expression, and divide each of them by the H. C. F.

156. It will be observed that the product of the H. C. F. and the L. C. M. of two expressions is equal to the product of the given expressions. For,

Let A and B denote the two expressions, and D their H. C. F.

Suppose $A=aD$, and $B=bD$;

Since D consists of all the factors common to A and B, a and b have no common factor.

$\therefore$ L. C. M. of a and b is ab.

Hence, the L. C. M. of aD and bD is abD.

Now, $A=aD$, and $B=bD$;

$\therefore AB=abD\times D$.

$\therefore \dfrac{AB}{D}=abD=$ the lowest common multiple. That is,

The L. C. M. *of two expressions can be found by dividing their product by their* H. C. F.

Or, *by dividing one of the expressions by the* H. C. F., *and multiplying the result by the other expression.*

157. To find the L. C. M. of *three* expressions, A, B, C. Find M, the L. C. M. of A and B; then the L. C. M. of M and C is the L. C. M. required.

Exercise LI.

Find the L. C. M. of:

1. $6x^2-x-2, \quad 21x^2-17x+2, \quad 14x^2+5x-1.$
2. $x^2-1, \quad x^2+2x-3, \quad 6x^2-x-2.$
3. $x^3-27, \quad x^2-15x+36, \quad x^3-3x^2-2x+6.$
4. $5x^2+19x-4, \quad 10x^2+13x-3.$
5. $12x^2+xy-6y^2, \quad 18x^2+18xy-20y^2.$
6. $x^4-2x^3+x, \quad 2x^4-2x^3-2x-2.$
7. $12x^2+2x-4, \quad 12x^2-42x-24, \quad 12x^2-28x-24.$
8. $x^3-6x^2+11x-6, \quad x^3-9x^2+26x-24,$ $x^3-8x^2+19x-12.$
9. $x^2-4a^2, \quad x^3+2ax^2+4a^2x+8a^3, \quad x^3-2ax^2+4a^2x-8a^3.$
10. $x^3+2x^2y-xy^2-2y^3, \quad x^3-2x^2y-xy^2+2y^3.$
11. $1+p+p^2, \quad 1-p+p^2, \quad 1+p^2+p^4.$
12. $(1-a), \quad (1-a)^2, \quad (1-a)^3.$
13. $(a+c)^2-b^2, \quad (a+b)^2-c^2, \quad (b+c)^2-a^2.$
14. $3c^3-3c^2y+cy^2-y^3, \quad 4c^3-c^2y-3cy^2.$
15. $m^3-8m+3, \quad m^6+3m^5+m+3.$
16. $20n^4+n^2-1, \quad 25n^4+5n^3-n-1.$
17. $b^4-2b^3+b^2-8b+8, \quad 4b^3-12b^2+9b-1.$
18. $2r^5-8r^4+12r^3-8r^2+2r, \quad 3r^5-6r^3+3r.$

CHAPTER VIII.

Fractions.

158. The expression $\frac{a}{b}$ is employed to indicate that a units are divided into b equal parts, and that *one* of these parts is taken;

or, that *one* unit is divided into b equal parts, and that a of these parts are taken.

159. The expression $\frac{a}{b}$ is called a **fraction.** a is the **numerator,** and b the **denominator.**

160. The numerator and denominator are called the **terms** of the fraction.

161. The denominator shows into how many equal parts the unit is divided, and therefore *names* the part; and the numerator shows how many of these parts are taken.

It will be observed that a letter written *above* the line in a fraction serves a very different purpose from that of a letter written *below* the line.

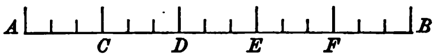
A letter written above the line denotes **number;**

A letter written below the line denotes **name.**

162. Every whole number may be written in the form of a fraction with unity for its denominator; thus, $a = \frac{a}{1}$.

To Reduce a Fraction to its Lowest Terms.

163. Let the line AB be divided into 5 equal parts, at the points C, D, E, F.



Then AF is $\frac{4}{5}$ of AB. (1)

Now let each of the parts be subdivided into 3 equal parts.

Then AB contains 15 of these subdivisions, and AF contains 12 of these subdivisions.

$\therefore$ AF is $\frac{12}{15}$ of AB. (2)

Comparing (1) and (2), it is evident that $\frac{4}{5} = \frac{12}{15}$.

In general:

If we suppose AB to be divided into b equal parts, and that AF contains a of these parts,

Then AF is $\frac{a}{b}$ of AB. (3)

Now, if we suppose each of the parts to be subdivided into c equal parts,

Then AB contains bc of these subdivisions, and AF contains ac of these subdivisions.

$\therefore$ AF is $\frac{ac}{bc}$ of AB. (4)

Comparing (3) and (4), it is evident that

$$\frac{a}{b} = \frac{ac}{bc}.$$

Since $\frac{ac}{bc}$ is obtained by multiplying by c both terms of the fraction $\frac{a}{b}$,

and, conversely, $\frac{a}{b}$ is obtained by dividing by c both terms of the fraction $\frac{ac}{bc}$, it follows that

I. If the numerator and denominator of a fraction be multiplied by the same number, the value of the fraction is not altered.

II. If the numerator and denominator be divided by the same number, the value of the fraction is not altered.

Hence, to reduce a fraction to lower terms,

Divide the numerator and denominator by any common factor.

164. A fraction is expressed in its lowest terms when both numerator and denominator are divided by their H. C. F.

Reduce the following fractions to their lowest terms:

(1) $$\frac{a^3-x^3}{a^2-x^2}=\frac{(a-x)(a^2+ax+x^2)}{(a-x)(a+x)}=\frac{a^2+ax+x^2}{a+x}.$$

(2) $$\frac{a^2+7a+10}{a^2+5a+6}=\frac{(a+5)(a+2)}{(a+3)(a+2)}=\frac{a+5}{a+3}.$$

(3) $$\frac{6x^2-5x-6}{8x^2-2x-15}=\frac{(2x-3)(3x+2)}{(2x-3)(4x+5)}=\frac{3x+2}{4x+5}.$$

(4) $$\frac{a^3-7a^2+16a-12}{3a^3-14a^2+16a}.$$

Since in Ex. (4) no common factor can be determined by inspection, it is necessary to find the H. C. F. of the numerator and denominator by the method of division.

Suppress the factor a of the denominator and proceed to divide:

$$
\begin{array}{r|r|l}
a^3 - 7a^2 + 16a - 12 & 3a^2 - 14a + 16 & a - 7 \\
3 & \underline{3a^2 - 6a} & \\
\overline{3a^3 - 21a^2 + 48a - 36} & -8a + 16 & 3a - 8 \\
\underline{3a^3 - 14a^2 + 16a\phantom{-36}} & \underline{-8a + 16} & \\
-7a^2 + 32a - 36 & & \\
\underline{3} & & \\
-21a^2 + 96a - 108 & & \\
\underline{-21a^2 + 98a - 112} & & \\
-2)\underline{-2a + 4} & & \\
a - 2 & &
\end{array}
$$

$\therefore$ the H. C. F. $= a - 2$.

Now, if $a^3 - 7a^2 + 16a - 12$ be divided by $a - 2$, the result is $a^2 - 5a + 6$; and if $3a^3 - 14a^2 + 16a$ be divided by $a - 2$, the result is $3a^2 - 8a$.

$$\therefore \frac{a^3 - 7a^2 + 16a - 12}{3a^3 - 14a^2 + 16a} = \frac{a^2 - 5a + 6}{3a^2 - 8a}.$$

165. When common factors cannot be determined by inspection, the H. C. F. must be found by the method of division.

Exercise LII.

Reduce to lowest terms:

1. $\dfrac{x^2 - 1}{4x(x + 1)}$.

2. $\dfrac{x^2 - 9x + 20}{x^2 - 7x + 12}$.

3. $\dfrac{x^2 - 2x - 3}{x^2 - 10x + 21}$.

4. $\dfrac{x^4 + x^2 + 1}{x^2 + x + 1}$.

5. $\dfrac{x^6 + 2x^3y^3 + y^6}{x^6 - y^6}$.

6. $\dfrac{a^3 + 1}{a^3 + 2a^2 + 2a + 1}$.

7. $\dfrac{a^2 - a - 20}{a^2 + a - 12}$.

8. $\dfrac{x^3 - 4x^2 + 9x - 10}{x^3 + 2x^2 - 3x + 20}$.

9. $\dfrac{x^3 - 5x^2 + 11x - 15}{x^3 - x^2 + 3x + 5}$.

10. $\dfrac{x^4 + x^3y + xy^3 - y^4}{x^4 - x^3y - xy^3 - y^4}$.

11. $\dfrac{a^3+4a^2-5}{a^3-3a+2}$.

12. $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}$.

13. $\dfrac{x^3-3x^2+4x-2}{x^3-x^2-2x+2}$.

14. $\dfrac{4x^2-12ax+9a^2}{8x^3-27a^3}$.

15. $\dfrac{15a^2+ab-2b^2}{9a^2+3ab-2b^2}$.

16. $\dfrac{a^2-b^2-2bc-c^2}{a^2+2ab+b^2-c^2}$.

17. $\dfrac{x^4-x^2-2x+2}{2x^3-x-1}$.

18. $\dfrac{x^3-6x^2+11x-6}{x^3-2x^2-x+2}$.

19. $\dfrac{6x^3-23x^2+16x-3}{6x^3-17x^2+11x-2}$.

20. $\dfrac{x^4-x^3-x+1}{x^4-2x^3-x^2-2x+1}$.

21. $\dfrac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^3}$.

22. $\dfrac{(a+b)^2}{a^2-ab-2b^2}$.

23. $\dfrac{3ab\,(a^2-b^2)}{4\,(a^2b-ab^2)^2}$.

24. $\dfrac{a^2+2ab+b^2-c^2}{a^2+ab-ac}$.

25. $\dfrac{6x^3-11x^2y+3xy^2}{6x^2y-5xy^2-6y^3}$.

26. $\dfrac{a^2-(b+c+d)^2}{(a-b)^2-(c+d)^2}$.

27. $\dfrac{6x^2-5x-6}{8x^2-2x-15}$.

28. $\dfrac{x^4+x^2y^2+y^4}{(x-y)(x^3-y^3)}$.

29. $\dfrac{x^6+y^6}{x^4-x^2y^2+y^4}$.

30. $\dfrac{(a^3+b^3)(a^2+ab+b^2)}{(a^3-b^3)(a^2-ab+b^2)}$.

To Reduce a Fraction to an Integral or Mixed Expression.

Change $\dfrac{x^3+1}{x-1}$ to a mixed expression.

$$(x^3+1)\div(x-1)=x^2+x+1+\frac{2}{x-1}.\quad \text{Hence,}$$

166. *If the degree of the numerator of a fraction equals or exceeds that of the denominator, the fraction may be changed to the form of a mixed or integral expression by dividing the numerator by the denominator.*

The quotient will be the integral expression, the remainder (if any) will be the numerator, and the divisor the denominator, of the fractional expression.

Exercise LIII.

Change to integral or mixed expressions:

1. $\dfrac{x^2-2x+1}{x-1}$.

2. $\dfrac{3x^2+2x+1}{x+4}$.

3. $\dfrac{3x^2+6x+5}{x+4}$.

4. $\dfrac{a^2-ax+x^2}{a+x}$.

5. $\dfrac{2x^2+5}{x-3}$.

6. $\dfrac{10a^2-17ax+10x^2}{5a-x}$.

7. $\dfrac{16(3x^2+1)}{4x-1}$.

8. $\dfrac{2x^2-5x-2}{x-4}$.

9. $\dfrac{a^2+b^2}{a-b}$.

10. $\dfrac{5x^3-x^2+5}{5x^2+4x-1}$.

To Reduce a Mixed Expression to the Form of a Fraction.

167. In arithmetic $5\frac{3}{4}$ means $5+\frac{3}{4}$.

But in algebra the fraction connected with the integral expression, as well as the integral expression, may be positive or negative; so that a mixed expression may occur in any one of the following forms:

$$n+\frac{a}{b};\quad n-\frac{a}{b};\quad -n+\frac{a}{b};\quad -n-\frac{a}{b}.$$

Change $n+\frac{a}{b}$ to a fractional form.

Since there are b *bths* in 1, in n there will be n times b *bths*, that is, nb *bths*, which, with the additional a *bths*, make $nb+a$ *bths*.

$$\therefore\ n+\frac{a}{b}=\frac{nb+a}{b}.$$

In like manner:

$$n-\frac{a}{b}=\frac{nb-a}{b};$$

$$-n+\frac{a}{b}=\frac{-nb+a}{b};$$

and $-n-\frac{a}{b}=\frac{-nb-a}{b}$. Hence,

168. To reduce a mixed expression to a fractional form,

Multiply the integral expression by the denominator, to the product annex the numerator, and under the result write the denominator.

169. It will be seen that the *sign* before the fraction is transferred to the *numerator* when the mixed expression is reduced to the fractional form, for the denominator shows only what *part* of the numerator is to be added or subtracted.

The dividing line has the force of a vinculum or parenthesis affecting the numerator; therefore if a *minus sign* precede the dividing line, and this line be removed, the *sign of every term of the numerator must be changed.* Thus,

$$n-\frac{a-b}{c}=\frac{cn-(a-b)}{c}=\frac{cn-a+b}{c}.$$

(1) Change to fractional form $x-1+\frac{x-1}{x}$.

$$x-1+\frac{x-1}{x}$$
$$=\frac{x^2-x+(x-1)}{x},$$
$$=\frac{x^2-x+x-1}{x},$$
$$=\frac{x^2-1}{x}.$$

(2) Change to fractional form $x-1-\frac{x-1}{x}$.

$$x-1-\frac{x-1}{x},$$
$$=\frac{x^2-x-(x-1)}{x},$$
$$=\frac{x^2-x-x+1}{x},$$
$$=\frac{x^2-2x+1}{x}.$$

Exercise LIV.

Change to fractional form:

1. $1-\frac{x-y}{x+y}$.

2. $1+\frac{x-y}{x+y}$.

3. $3x-\frac{1+2x^2}{x}$.

4. $a-x+\frac{a^2+x^2}{a-x}$.

5. $5a-2b-\frac{3a^2-4b^2}{5a-6b}$.

6. $a+b-\frac{a^2+b^2}{a+b}$.

7. $7a-\frac{2-3a+4a^2}{5-6a}$.

8. $3x-\frac{5ax-3}{2a}$.

9. $\dfrac{a+b}{a-b}+1.$

10. $\dfrac{a-b}{a+b}-1.$

11. $\dfrac{2x^2}{x+y}-(x+y).$

12. $\dfrac{5a-12x}{4}+6a+3x.$

13. $a-1+\dfrac{1}{a+1}.$

14. $x+5-\dfrac{2x-15}{x-3}.$

15. $2a-b-\dfrac{2ab}{a+b}.$

16. $3x-10+\dfrac{41}{x+4}.$

17. $x^2+x+1+\dfrac{2}{x-1}.$

18. $x^2-3x-\dfrac{3x(3-x)}{x-2}.$

19. $a^2-2ax+4x^2-\dfrac{6x^3}{a+2x}.$

20. $x-a+y+\dfrac{a^2-ay+y^2}{x+a}.$

Lowest Common Denominator.

170. To reduce fractions to equivalent fractions having the lowest common denominator:

Reduce $\dfrac{3x}{4a^2}$, $\dfrac{2y}{3a}$, and $\dfrac{5}{6a^3}$ to equivalent fractions having the lowest common denominator.

The L. C. M. of $4a^2$, $3a$, and $6a^3 = 12a^3$.

If both terms of $\dfrac{3x}{4a^2}$ be multiplied by $3a$, the value of the fraction will not be altered, but the form will be changed to $\dfrac{9ax}{12a^3}$; if both terms of $\dfrac{2y}{3a}$ be multiplied by $4a^2$, the equivalent fraction $\dfrac{8a^2y}{12a^3}$ is obtained; and, if both terms of $\dfrac{5}{6a^3}$ be multiplied by 2, the equivalent fraction $\dfrac{10}{12a^3}$ is obtained.

Hence, $\dfrac{3x}{4a^2}$, $\dfrac{2y}{3a}$, $\dfrac{5}{6a^3}$

are equal to $\dfrac{9ax}{12a^3}$, $\dfrac{8a^2y}{12a^3}$, $\dfrac{10}{12a^3}$, respectively.

The multipliers $3a$, $4a^2$, and 2, are obtained by dividing $12a^3$, the L. C. M. of the denominators, by the respective denominators of the given fractions.

171. Therefore, to reduce fractions to equivalent fractions having the lowest common denominator,

Find the L. C. M. *of the denominators.*

Divide the L. C. M. *by the denominator of each fraction.*

Multiply the first numerator by the first quotient, the second by the second quotient, and so on.

The products will be the numerators of the equivalent fractions.

The L. C. M. *of the given denominators will be the denominator of each of the equivalent fractions.*

Exercise LV.

Reduce to equivalent fractions with the lowest common denominator:

1. $\dfrac{3x-7}{6}, \dfrac{4x-9}{18}$.

2. $\dfrac{2x-4y}{5x^2}, \dfrac{3x-8y}{10x}$.

3. $\dfrac{4a-5c}{5ac}, \dfrac{3a-2c}{12a^2c}$.

4. $\dfrac{5}{1-x}, \dfrac{6}{1-x^2}$.

5. $\dfrac{1}{(a-b)(b-c)}, \dfrac{1}{(a-b)(a-c)}$.

6. $\dfrac{4x^2}{3(a+b)}, \dfrac{xy}{6(a^2-b^2)}$.

7. $\dfrac{8x+2}{x-2}, \dfrac{2x-1}{3x-6}, \dfrac{3x+2}{5x-10}$.

8. $\dfrac{a-bm}{mx}, 1, \dfrac{c-bn}{nx}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

172. To add fractions:

Reduce the fractions to equivalent fractions having the lowest common denominator.

Add the numerators of the equivalent fractions.

Write the result over the lowest common denominator.

173. To subtract one fraction from another:

Reduce the fractions to equivalent fractions having the lowest common denominator.

Subtract the numerator of the subtrahend from the numerator of the minuend.

Write the result over the lowest common denominator.

(1) Simplify, $\dfrac{4x+7}{5}+\dfrac{3x-4}{15}.$

The lowest common denominator (L. C. D.) $=15$.

The multipliers are 3 and 1 respectively.

$$\begin{array}{rl} 12x+21 &= \text{1st numerator,} \\ 3x-4 &= \text{2d numerator,} \\ \hline 15x+17 &= \text{sum of numerators.} \end{array}$$

$$\therefore \frac{4x+7}{5}+\frac{3x-4}{15}=\frac{15x+17}{15}.$$

(2) Simplify, $\dfrac{3a-4b}{7}-\dfrac{2a-b+c}{3}+\dfrac{13a-4c}{12}.$

The L. C. D. $=84$.

The multipliers are 12, 28, and 7 respectively.

$$\begin{array}{rl} 36a-48b &= \text{1st numerator,} \\ -56a+28b-28c &= \text{2d numerator,} \\ 91a-28c &= \text{3d numerator.} \\ \hline 71a-20b-56c &= \text{sum of numerators.} \end{array}$$

$$\therefore \frac{3a-4b}{7}-\frac{2a-b+c}{3}+\frac{13a-4c}{12}=\frac{71a-20b-56c}{84}.$$

Since the *minus sign* precedes the second fraction, the signs of all the terms of the numerator of this fraction are changed after being multiplied by 28.

Exercise LVI.

Simplify:

1. $\dfrac{3x-2y}{5x}+\dfrac{5x-7y}{10x}+\dfrac{8x+2y}{25}.$

2. $\dfrac{4x^2-7y^2}{3x^2}+\dfrac{3x-8y}{6x}+\dfrac{5-2y}{12}.$

3. $\dfrac{4a^2+5b^2}{2b^2}+\dfrac{3a+2b}{5b}+\dfrac{7-2a}{9}.$

4. $\dfrac{4x+5}{3}-\dfrac{3x-7}{5x}+\dfrac{9}{12x^2}.$

5. $\dfrac{4x-3y}{7}+\dfrac{3x+7y}{14}-\dfrac{5x-2y}{21}+\dfrac{9x+2y}{42}.$

6. $\dfrac{3xy-4}{x^2y^2}-\dfrac{5y^2+7}{xy^3}-\dfrac{6x^2-11}{x^3y}.$

7. $\dfrac{a^2-2ac+c^2}{a^2c^2}-\dfrac{b^2-2bc+c^2}{b^2c^2}.$

8. $\dfrac{5a^3-2}{8a^2}-\dfrac{3a^2-a}{8}.$

9. $\dfrac{a-b}{c}+\dfrac{b-c}{a}+\dfrac{c-a}{b}+\dfrac{ab^2+bc^2+ca^2}{abc}.$

10. $\dfrac{1}{2x^2y}-\dfrac{1}{6y^2z}-\dfrac{1}{2xz^2}+\dfrac{2x-z}{4x^2z^2}+\dfrac{y-2z}{4x^2yz}.$

Simplify $\dfrac{x-y}{x+y}+\dfrac{x+y}{x-y}$.

The L. C. D. $=x^2-y^2$.

The multipliers are $x-y$ and $x+y$ respectively.

$$\begin{array}{rl} x^2-2xy+\ y^2 = & \text{1st numerator,} \\ x^2+2xy+\ y^2 = & \text{2d numerator.} \\ \hline 2x^2 \qquad +2y^2 = & \text{sum of numerators.} \\ \text{or, } 2(x^2+y^2) \qquad = & \text{" " "} \end{array}$$

$$\therefore \frac{x-y}{x+y}+\frac{x+y}{x-y}=\frac{2(x^2+y^2)}{x^2-y^2}.$$

Exercise LVII.

Simplify:

1. $\dfrac{1}{x-6}+\dfrac{1}{x+5}$.
2. $\dfrac{1}{x-7}-\dfrac{1}{x-3}$.
3. $\dfrac{1}{1+x}+\dfrac{1}{1-x}$.
4. $\dfrac{1}{1-x}-\dfrac{2}{1-x^2}$.
5. $\dfrac{1}{x-y}+\dfrac{x}{(x-y)^2}$.
6. $\dfrac{1}{2a(a+x)}+\dfrac{1}{2a(a-x)}$.
7. $\dfrac{a}{(a+b)b}-\dfrac{b}{(a-b)a}$.
8. $\dfrac{5}{2x(x-1)}-\dfrac{3}{4x(x-2)}$.
9. $\dfrac{1+x}{1+x+x^2}-\dfrac{1-x}{1-x+x^2}$.
10. $\dfrac{2ax-3by}{2xy(x-y)}-\dfrac{2ax+3by}{2xy(x+y)}$.

(1) Simplify $\dfrac{2a+b}{a-b}-\dfrac{2a-b}{a+b}-\dfrac{6ab}{a^2-b^2}$.

The L. C. D. $=(a-b)(a+b)$.

The multipliers are $a+b$, $a-b$, and 1, respectively.

$$\begin{array}{rl}
2a^2+3ab+b^2 & = \text{1st numerator,} \\
-2a^2+3ab-b^2 & = \text{2d numerator,} \\
-6ab \quad & = \text{3d numerator.} \\
\hline
0 \qquad & = \text{sum of numerators.}
\end{array}$$

$$\therefore \frac{2a+b}{a-b}-\frac{2a-b}{a+b}-\frac{6ab}{a^2-b^2}=0.$$

(2) Simplify $\dfrac{y^2}{x^2-y^2}-\dfrac{x-y}{x+y}+1+\dfrac{2xy}{x^2+y^2}$.

The L. C. D. $=(x+y)(x-y)(x^2+y^2)$.

The multipliers are x^2+y^2, $(x-y)(x^2+y^2)$, $(x+y)(x-y)$, (x^2+y^2), $(x+y)(x-y)$, respectively.

$$\begin{array}{rl}
x^2y^2 \qquad\qquad +y^4 & = \text{1st numerator,} \\
-x^4+2x^3y-2x^2y^2+2xy^3-y^4 & = \text{2d numerator,} \\
x^4 \qquad\qquad\qquad\qquad -y^4 & = \text{3d numerator,} \\
2x^3y \qquad\qquad -2xy^3 \qquad & = \text{4th numerator.} \\
\hline
4x^3y-x^2y^2 \qquad\qquad -y^4 & = \text{sum of numerators.}
\end{array}$$

$$\therefore \text{Sum of fractions}=\frac{4x^3y-x^2y^2-y^4}{x^4-y^4}.$$

Exercise LVIII.

Simplify:

1. $\dfrac{1}{1+a}+\dfrac{1}{1-a}+\dfrac{2a}{1-a^2}$.

2. $\dfrac{1}{1-x}-\dfrac{1}{1+x}+\dfrac{2x}{1+x^2}$.

3. $\dfrac{x}{1-x}-\dfrac{x^2}{1-x}+\dfrac{x}{1+x^2}$.

4. $\dfrac{x}{y}+\dfrac{y}{x+y}+\dfrac{x^2}{x^2+xy}$.

5. $\dfrac{x-1}{x-2}+\dfrac{x-2}{x-3}+\dfrac{x-3}{x-4}$.

6. $\dfrac{3}{x-a}+\dfrac{4a}{(x-a)^2}-\dfrac{5a^2}{(x-a)^3}$.

7. $\dfrac{1}{x-1}-\dfrac{1}{x+2}-\dfrac{3}{(x+1)(x+2)}.$

8. $\dfrac{a-b}{(b+c)(c+a)}+\dfrac{b-c}{(c+a)(a+b)}+\dfrac{c-a}{(a+b)(b+c)}.$

9. $\dfrac{x-a}{x-b}+\dfrac{x-b}{x-a}-\dfrac{(a-b)^2}{(x-a)(x-b)}.$

10. $\dfrac{x+y}{y}-\dfrac{2x}{x+y}+\dfrac{x^2y-x^3}{y(x^2-y^2)}.$

11. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}+\dfrac{c+a}{(a-b)(b-c)}.$

12. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ac}{(b+a)(b+c)}+\dfrac{c^2+ab}{(c+b)(c+a)}.$

13. $\dfrac{a}{a-x}-\dfrac{x}{a+2x}-\dfrac{a^2+x^2}{(a-x)(a+2x)}.$

14. $\dfrac{3}{(a-b)(b-c)}-\dfrac{4}{(a-b)(a-c)}+\dfrac{6}{(a-c)(b-c)}.$

15. $\dfrac{x-2y}{x(x-y)}-\dfrac{2x+y}{y(x+y)}-\dfrac{2x}{x^2-y^2}.$

16. $\dfrac{a-b}{x(a+b)}-\dfrac{a-b}{y(a+b)}-\dfrac{(a-b)(x+y)}{xy(a+b)}.$

17. $\dfrac{3x}{(x+y)^2}-\dfrac{x+2y}{x^2-y^2}+\dfrac{3y}{(x-y)^2}.$

18. $\dfrac{a-c}{(a+b)^2-c^2}-\dfrac{a-b}{(a+c)^2-b^2}.$

19. $\dfrac{a+b}{ax+by}-\dfrac{a-b}{ax-by}+\dfrac{ab(x-y)}{a^2x^2-b^2y^2}.$

174. Since $\quad \dfrac{ab}{b} = a$, and $\dfrac{-ab}{-b} = a$,

it is evident that if the signs of both numerator and denominator be changed, the value of the fraction is not altered.

Again, $\dfrac{a-b}{c-d} = \dfrac{-(a-b)}{-(c-d)} = \dfrac{-a+b}{-c+d} = \dfrac{b-a}{d-c}.$

Therefore, if the numerator or denominator be a compound expression, or if both be compound expressions, the sign of every term in the denominator may be changed, provided the sign of every term in the numerator be also changed.

Since the change of the sign before the fraction is equivalent to the change of the sign before every term of the numerator of the fraction, *the sign before every term of the denominator may be changed, provided the sign before the fraction be changed.*

Since, also, the product of $+a$ multiplied by $+b$ is ab, and the product of $-a$ multiplied by $-b$ is ab, the signs of *two factors*, or of *any even number of factors*, of the denominator of a fraction may be changed without altering the value of the fraction.

By the application of these principles, fractions may often be changed to a more simple form for addition or subtraction.

(1) Simplify $\dfrac{2}{x} - \dfrac{3}{2x-1} + \dfrac{2x-3}{1-4x^2}.$

Change the signs before the terms of the denominator of the third fraction, and change the sign before the fraction.

The result is,

$$\frac{2}{x} - \frac{3}{2x-1} - \frac{2x-3}{4x^2-1},$$

in which the several denominators are written in symmetrical form.

The L. C. D. $= x(2x-1)(2x+1)$.

$$\begin{array}{rl} 8x^2 - 2 & = \text{1st numerator,} \\ -6x^2 - 3x & = \text{2d numerator,} \\ -2x^2 + 3x & = \text{3d numerator.} \\ \hline -2 & = \text{sum of numerators.} \end{array}$$

$$\therefore \text{ Sum of the fractions} = \frac{-2}{x(2x-1)(2x+1)}.$$

(2) Simplify

$$\frac{1}{a(a-b)(a-c)} + \frac{1}{b(b-a)(b-c)} + \frac{1}{c(c-a)(c-b)}.$$

Change the sign of the factor $(b-a)$ in the denominator of the second fraction, and change the sign before the fraction.

Then change the signs of the factors $(c-a)$ and $(c-b)$ in the denominator of the third fraction.

The result is,

$$\frac{1}{a(a-b)(a-c)} - \frac{1}{b(a-b)(b-c)} + \frac{1}{c(a-c)(b-c)},$$

in which the factors of the several denominators are written in symmetrical form.

The L. C. D. $= abc(a-b)(a-c)(b-c)$.

$$\begin{array}{rll} bc(b-c) & = b^2c - bc^2 & = \text{1st numerator,} \\ -ac(a-c) & = -a^2c + ac^2 & = \text{2d numerator,} \\ ab(a-b) & = a^2b - ab^2 & = \text{3d numerator.} \\ \hline \multicolumn{2}{r}{a^2b - a^2c - ab^2 + ac^2 + b^2c - bc^2} & = \text{sum of numerators.} \end{array}$$

$$\begin{aligned} &= a^2(b-c) - a(b^2-c^2) + bc(b-c), \\ &= [a^2 - a(b+c) + bc][b-c], \\ &= [a^2 - ab - ac + bc][b-c], \\ &= [(a^2-ac) - (ab-bc)][b-c], \\ &= [a(a-c) - b(a-c)][b-c], \\ &= (a-b)(a-c)(b-c). \end{aligned}$$

$$\therefore \text{ Sum of the fractions} = \frac{(a-b)(a-c)(b-c)}{abc\,(a-b)(a-c)(b-c)}$$

$$= \frac{1}{abc}.$$

Exercise LIX.

Simplify:

1. $\dfrac{x}{x-y} + \dfrac{x-y}{y-x}$.

2. $\dfrac{3+2x}{2-x} + \dfrac{3x-2}{2+x} + \dfrac{16x-x^2}{x^2-4}$.

3. $\dfrac{x^2}{x^2-1} + \dfrac{x}{x+1} - \dfrac{x}{1-x}$.

4. $\dfrac{4}{3-3y^2} + \dfrac{1}{2-2y} + \dfrac{1}{6y+6}$.

5. $\dfrac{1}{(2-m)(3-m)} - \dfrac{2}{(m-1)(m-3)} + \dfrac{1}{(m-1)(m-2)}$

6. $\dfrac{1}{(b-a)(x+a)} + \dfrac{1}{(a-b)(x+b)}$.

7. $\dfrac{a^2+b^2}{a^2-b^2} + \dfrac{2ab^2}{b^3-a^3} + \dfrac{2a^2b}{a^3+b^3}$.

8. $\dfrac{b-a}{x-b} - \dfrac{a-2b}{b+x} - \dfrac{3x(a-b)}{b^2-x^2}$.

9. $\dfrac{3+2x}{2-x} - \dfrac{2-3x}{2+x} + \dfrac{16x-x^2}{x^2-4}$.

10. $\dfrac{3}{1-2x} - \dfrac{7}{1+2x} - \dfrac{4-20x}{4x^2-1}$.

11. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(b-a)(a-c)}+\dfrac{c+a}{(a-b)(b-c)}.$

12. $\dfrac{a^2-bc}{(a-b)(a-c)}+\dfrac{b^2+ac}{(b+c)(b-a)}+\dfrac{c^2+ab}{(c-a)(c+b)}.$

13. $\dfrac{y+z}{(x-y)(x-z)}+\dfrac{z+x}{(y-x)(y-z)}+\dfrac{x+y}{(z-x)(z-y)}.$

14. $\dfrac{3}{(a-b)(b-c)}-\dfrac{4}{(b-a)(c-a)}-\dfrac{6}{(a-c)(c-b)}.$

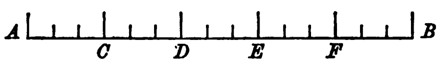
15. $\dfrac{1}{x(x-y)(x-z)}+\dfrac{1}{y(y-x)(y-z)}-\dfrac{1}{xyz}.$

Multiplication of Fractions.

175. Hitherto in fractions, equal parts of one or more *units* have been taken. But it is often necessary to take equal parts of *fractions of units*.

Suppose it is required to take $\frac{2}{3}$ of $\frac{4}{5}$ of a unit.

Let the line AB represent the unit of length.



Suppose AB divided into 5 equal parts, at C, D, E, and F, and each of these parts to be subdivided into 3 equal subdivisions.

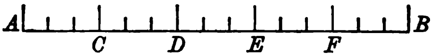
Then one of the parts, as AC, will contain 3 of these subdivisions, and the whole line AB will contain 15 of these subdivisions.

That is, $\frac{1}{3}$ of $\frac{1}{5}$ of the line will be $\frac{1}{15}$ of the line;

$\frac{1}{3}$ of $\frac{4}{5}$ will be $\frac{1}{15}+\frac{1}{15}+\frac{1}{15}+\frac{1}{15}$, or $\frac{4}{15}$, of the line; and

$\frac{2}{3}$ of $\frac{4}{5}$ will be *twice* $\frac{4}{15}$, or $\frac{8}{15}$, of the line.

Suppose it is required to take $\frac{c}{d}$ of $\frac{a}{b}$ of the line AB.



Let the line AB be divided into b equal parts, and let each of these parts be subdivided into d equal subdivisions.

Then the whole line will contain bd of these subdivisions, and one of these subdivisions will be $\frac{1}{bd}$ of the line.

If one of the subdivisions be taken from each of a parts, they will together be $\frac{a}{bd}$ of the line. That is,

$$\frac{1}{d} \text{ of } \frac{a}{b} = \frac{1}{bd} + \frac{1}{bd} + \frac{1}{bd} \cdots\cdots \text{ taken } a \text{ times}, = \frac{a}{bd},$$

and $\frac{c}{d}$ of $\frac{a}{b}$ will be c *times* $\frac{a}{bd}$, or $\frac{ac}{bd}$ of the line.

Therefore, to find a fraction of a fraction,

Find the product of the numerators for the numerator of the product, and of the denominators for the denominator of the product.

176. Now, $\frac{c}{d} \times \frac{a}{b}$ means $\frac{c}{d}$ of $\frac{a}{b}$.

Therefore, to find the product of two fractions,

Find the product of the numerators for the numerator of the product, and of the denominators for the denominator of the product.

The same rule will hold when more than two fractions are taken.

If a factor exist in both a numerator and a denominator, it may be cancelled; for the cancelling of a common factor *before* the multiplication is evidently equivalent to cancelling it *after* the multiplication; and this may be done by § 163.

Division of Fractions.

177. Multiplying by the reciprocal of a number is equivalent to dividing by the number. Thus, multiplying by $\frac{1}{4}$ is equivalent to dividing by 4.

The reciprocal of a fraction is the fraction with its terms interchanged.

Thus, the reciprocal of $\frac{2}{3}$ is $\frac{3}{2}$, for $\frac{3}{2}\times\frac{2}{3}=1$. § 42.

Therefore, to divide by a fraction,

Interchange the terms of the fraction and multiply by the resulting fraction. Thus,

(1) $$\frac{2a}{3x^2}\div\frac{1}{3x}=\frac{2a}{3x^2}\times\frac{3x}{1}=\frac{2a}{x}.$$

The common factor cancelled is $3x$.

(2) $$\frac{14x^2}{27y^2}\div\frac{7x}{9y}=\frac{14x^2}{27y^2}\times\frac{9y}{7x}=\frac{2x}{3y}.$$

The common factors cancelled are $9y$ and $7x$.

(3) $$\frac{ax}{(a-x)^2}\div\frac{ab}{a^2-x^2}=\frac{ax}{(a-x)(a-x)}\times\frac{(a+x)(a-x)}{ab}$$
$$=\frac{x(a+x)}{b(a-x)}.$$

The common factors cancelled are a and $a-x$.

If the divisor be an integral expression, it may be changed to the fractional form. § 162.

Exercise LX.

1. $\dfrac{a}{bx}\times\dfrac{cx}{d}$.

2. $\dfrac{2x}{a}\times\dfrac{3ab}{c}\times\dfrac{3ac}{2b}$.

3. $\dfrac{3p}{2p-2}\div\dfrac{2p}{p-1}$.

4. $\dfrac{8x^4y}{15ab^3}\div\dfrac{2x^3}{3ab^2}$.

5. $\frac{8a^2b^3}{45x^2y} \times \frac{15xy^2}{24a^3b^2}$.

8. $\frac{9m^2n^2}{8p^3q^3} \times \frac{5p^2q}{2xy} \times \frac{24x^2y^2}{90mn}$.

6. $\frac{9x^2y^2z}{10a^2b^2c} \times -\frac{20a^3b^2c}{18xy^2z}$.

9. $\frac{25k^3m^2}{14n^2q^2} \times \frac{70n^3q}{75p^2m} \times \frac{3pm}{4k^2n}$.

7. $\frac{3x^2y}{4xz^2} \times \frac{5y^2z}{6xy} \times -\frac{12x^2}{2xy^2}$.

10. $\frac{a-b}{a^2+ab} \times \frac{a^2-b^2}{a^2-ab}$.

11. $\frac{a^2+b^2}{a^2-b^2} \div \frac{a-b}{a+b}$.

12. $\frac{x^2+x-2}{x^2-7x} \times \frac{x^2-13x+42}{x^2+2x}$.

13. $\frac{x^2-11x+30}{x^2-6x+9} \times \frac{x^2-3x}{x^2-5x}$.

14. $\frac{a^3-x^3}{a^3+x^3} \times \frac{(a+x)^2}{(a-x)^2}$.

15. $\frac{2a(x^2-y^2)^2}{cx} \times \frac{x^3}{(x-y)(x+y)^2}$.

16. $\frac{a^2+2ab}{a^2+4b^2} \times \frac{ab-2b^2}{a^2-4b^2}$.

18. $\frac{x^2+xy}{x-y} \times \frac{(x-y)^2}{x^4-y^4}$.

17. $\frac{x^2-4}{x^2+5x} \times \frac{x^2-25}{x^2+2x}$.

19. $\frac{m^2-n^2}{c^3+d^3} \div \frac{n-m}{c+d}$.

20. $\frac{a^2-4a+3}{a^2-5a+4} \times \frac{a^2-9a+20}{a^2-10a+21} \times \frac{a^2-7a}{a^2-5a}$.

21. $\frac{b^2-7b+6}{b^2+3b-4} \times \frac{b^2+10b+24}{b^2-14b+48} \div \frac{b^2+6b}{b^3-8b^2}$.

22. $\frac{x^2-y^2}{x^2-3xy+2y^2} \times \frac{xy-2y^2}{x^2+xy} \times \frac{x^2-xy}{(x-y)^2}$.

23. $\frac{a^3-3a^2b+3ab^2-b^3}{a^2-b^2} \div \frac{2ab-2b^2}{3} \times \frac{a^2+ab}{a-b}$.

24. $\dfrac{(a+b)^2-c^2}{a^2-(b-c)^2} \div \dfrac{c^2-(a+b)^2}{c^2-(a-b)^2}.$

25. $\dfrac{(x-a)^2-b^2}{(x-b)^2-a^2} \times \dfrac{x^2-(b-a)^2}{x^2-(a-b)^2}.$

26. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2} \div \dfrac{(a-c)^2-(d-b)^2}{(a-b)^2-(d-c)^2}.$

27. $\dfrac{x^2-2xy+y^2-z^2}{x^2+2xy+y^2-z^2} \times \dfrac{x+y-z}{x-y+z}.$

Complex Fractions.

178. A **complex fraction** is one which has a fraction in the numerator or in the denominator, or in both.

179. A fraction may be regarded as the *quotient* of the numerator divided by the denominator.

This is the simplest meaning of a complex fraction.

Therefore, to simplify a complex fraction,

Divide the numerator by the denominator.

(1) Simplify $\dfrac{\frac{1}{3}}{\frac{1}{2}}$.

$$\frac{\frac{1}{3}}{\frac{1}{2}} = \tfrac{1}{3} \div \tfrac{1}{2} = \tfrac{1}{3} \times \tfrac{2}{1} = \tfrac{2}{3}.$$

(2) Simplify $\dfrac{2\frac{2}{3}}{5\frac{7}{8}}$.

$$\frac{2\frac{2}{3}}{5\frac{7}{8}} = \frac{\frac{8}{3}}{\frac{47}{8}} = \tfrac{8}{3} \div \tfrac{47}{8} = \tfrac{8}{3} \times \tfrac{8}{47} = \tfrac{64}{141}.$$

(3) Simplify $\dfrac{3x}{x-\frac{1}{4}}$.

$$\frac{3x}{x-\frac{1}{4}} = \frac{3x}{\frac{4x-1}{4}} = \frac{3x}{1} \div \frac{4x-1}{4} = \frac{3x}{1} \times \frac{4}{4x-1}$$
$$= \frac{12x}{4x-1}.$$

It is often shorter to multiply both terms of the fraction by the L. C. D. of the fractions contained in the numerator and denominator.

Thus, in (1), multiply both terms by 6; in (2), both terms by 24; in (3), by 4. The results obtained are $\frac{2}{3}$, $\frac{64}{141}$, $\frac{12x}{4x-1}$, respectively.

(4) Simplify $\dfrac{x}{1-\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}}$.

$$\frac{x}{1-\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}} = \frac{x}{1-\dfrac{x(1-x+x^2)}{(1+x)(1-x+x^2)+x}}$$

$$= \frac{x}{1-\dfrac{x-x^2+x^3}{1+x+x^3}}$$

$$= \frac{x(1+x+x^3)}{1+x+x^3-(x-x^2+x^3)}$$

$$= \frac{x+x^2+x^4}{1+x^2}.$$

The expression $\dfrac{x}{1+x+\dfrac{x}{1-x+x^2}}$ is reduced to the form $\dfrac{x(1-x+x^2)}{(1+x)(1-x+x^2)+x}$, which $=\dfrac{x-x^2+x^3}{1+x+x^3}$.

The expression $\dfrac{x}{1-\dfrac{x-x^2+x^3}{1+x+x^3}}$ is reduced to the form $\dfrac{x(1+x+x^3)}{1+x+x^3-(x-x^2+x^3)}$, which $=\dfrac{x+x^2+x^4}{1+x^2}$.

Exercise LXI.

Simplify:

1. $$\frac{\dfrac{3x}{2}+\dfrac{x-1}{3}}{\dfrac{13}{6}(x+1)-\dfrac{x}{3}-2\frac{1}{2}}$$

2. $$\frac{x-1+\dfrac{6}{x-6}}{x-2+\dfrac{3}{x-6}}$$

3. $$\frac{3}{x+1}-\frac{2x-1}{x^2+\dfrac{x}{2}-\dfrac{1}{2}}$$

4. $$\frac{x-a}{x-\dfrac{(x-b)(x-c)}{x+a}}$$

5. $$\frac{\left(\dfrac{a}{x}-\dfrac{x}{a}\right)\left(\dfrac{a}{x}+\dfrac{x}{a}\right)}{1-\dfrac{x-a}{x+a}}$$

6. $$\frac{\dfrac{1}{x-y}-\dfrac{x}{x^2-y^2}}{\dfrac{x}{xy+y^2}-\dfrac{y}{x^2+xy}}$$

7. $$\frac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x+1}}{\dfrac{x+1}{x-1}-\dfrac{x-1}{x+1}}$$

8. $$1-\frac{1}{1+\dfrac{1}{x}}$$

9. $$1+\frac{x}{1+x+\dfrac{2x^2}{1-x}}$$

10. $$\frac{1}{1-\dfrac{1}{1+\dfrac{1}{x}}}$$

11. $$\frac{1}{1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}}$$

12. $$\frac{\left(\dfrac{a}{x}+\dfrac{x}{a}-2\right)\left(\dfrac{a}{x}+\dfrac{x}{a}+2\right)}{\left(\dfrac{a}{x}-\dfrac{x}{a}\right)^2}$$

13. $$\frac{\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{2x}{x+y}\left\{\dfrac{xy-x^2}{(x-y)^2}+\dfrac{x+y}{x-y}\right\}}{x-y}$$

14. $$\frac{\dfrac{(x^2-y^2)(2x^2-2xy)}{4(x-y)^2}}{\dfrac{xy}{x+y}}$$

15. $$\frac{\dfrac{ab}{x^2+(a+b)x+ab}-\dfrac{ac}{x^2+(a+c)x+ac}}{\dfrac{b-c}{x^2+(b+c)x+bc}}$$

16. $$\frac{x}{1+\dfrac{1}{x}}+1-\frac{1}{x+1}$$

17. $$\frac{\dfrac{a+b}{b}+\dfrac{b}{a+b}}{\dfrac{1}{a}+\dfrac{1}{b}}$$

18. $$\frac{2m-3+\dfrac{1}{m}}{\dfrac{2m-1}{m}}$$

19. $$\frac{\dfrac{1}{ab}+\dfrac{1}{ac}+\dfrac{1}{bc}}{\dfrac{a^2-(b+c)^2}{ab}}$$

20. $$\frac{3}{1+\dfrac{3}{1+\dfrac{3}{1-x}}}$$

Exercise LXII.

MISCELLANEOUS EXAMPLES.

1. Simplify $\dfrac{x^4-9x^3+7x^2+9x-8}{x^4+7x^3-9x^2-7x+8}$.

2. Find the value of $\dfrac{a^2+b^2-c^2+2ab}{a^2-b^2-c^2+2bc}$ when $a=4$, $b=\frac{1}{2}$, $c=1$.

3. Find the value of $3a^2+\dfrac{2ab^2}{c}-\dfrac{c^3}{b^2}$ when $a=4$, $b=\frac{1}{2}$, $c=1$.

4. Simplify $\dfrac{2}{(x^2-1)^2}-\dfrac{1}{2x^2-4x+2}-\dfrac{1}{1-x^2}$.

5. Simplify $\left(\dfrac{x}{1+\dfrac{1}{x}}+1-\dfrac{1}{x+1}\right)\div\left(\dfrac{x}{1-\dfrac{1}{x}}-x-\dfrac{1}{x-1}\right)$.

6. Find the value of $\left(\dfrac{x-a}{x-b}\right)^3-\dfrac{x-2a+b}{x+a-2b}$ when $x=\dfrac{a+b}{2}$.

7. Simplify $\left\{\dfrac{a+b}{2(a-b)}-\dfrac{a-b}{2(a+b)}+\dfrac{2b^2}{a^2-b^2}\right\}\dfrac{a-b}{2b}$.

8. Simplify $\left(\frac{x^2+y^2}{x^2-y^2}-\frac{x^2-y^2}{x^2+y^2}\right)\div\left(\frac{x+y}{x-y}-\frac{x-y}{x+y}\right).$

9. Simplify

$$\left(\frac{x^2}{y^2}-1\right)\left(\frac{x}{x-y}-1\right)+\left(\frac{x^3}{y^3}-1\right)\left(\frac{x^2+xy}{x^2+xy+y^2}-1\right).$$

10. Simplify

$$\left(\frac{a^2-ab}{a^3-b^3}\right)\left(\frac{a^2+ab+b^2}{a+b}\right)+\left(\frac{2a^3}{a^3+b^3}-1\right)\left(1-\frac{2ab}{a^2+ab+b^2}\right).$$

11. Simplify $\dfrac{1+\dfrac{a-x}{a+x}}{1-\dfrac{a-x}{a+x}}\div\dfrac{1+\dfrac{a^2-x^2}{a^2+x^2}}{1-\dfrac{a^2-x^2}{a^2+x^2}}.$

12. Divide $x^3+\frac{1}{x^3}-3\left(\frac{1}{x^2}-x^2\right)+4\left(x+\frac{1}{x}\right)$ by $x+\frac{1}{x}.$

13. Simplify $\dfrac{1-\dfrac{2xy}{(x+y)^2}}{1+\dfrac{2xy}{(x-y)^2}}\div\left\{\dfrac{1-\dfrac{y}{x}}{1+\dfrac{y}{x}}\right\}^2.$

14. Find the value of $\frac{x+2a}{2b-x}+\frac{x-2a}{2b+x}-\frac{4ab}{4b^2-x^2}$ when $x=\frac{ab}{a+b}.$

15. Find the value of $\frac{x+y-1}{x-y+1}$ when $x=\frac{a+1}{ab+1}$ and $y=\frac{ab+a}{ab+1}.$

16. Simplify

$$\frac{1}{a(a-b)(a-c)}+\frac{1}{b(b-c)(b-a)}+\frac{1}{c(c-a)(c-b)}.$$

17. Simplify $\dfrac{3abc}{bc+ca-ab}-\dfrac{\dfrac{a-1}{a}+\dfrac{b-1}{b}+\dfrac{c-1}{c}}{\dfrac{1}{a}+\dfrac{1}{b}-\dfrac{1}{c}}.$

18. Simplify $\dfrac{\dfrac{m^2+n^2}{n}-m}{\dfrac{1}{n}-\dfrac{1}{m}} \times \dfrac{m^2-n^2}{m^3+n^3}$

19. Simplify $\dfrac{\dfrac{1}{a}+\dfrac{1}{b+c}}{\dfrac{1}{a}-\dfrac{1}{b+c}}\left\{1+\dfrac{b^2+c^2-a^2}{2bc}\right\}$

20. Simplify $3a-[b+\{2a-(b-c)\}]+\dfrac{1}{2}+\dfrac{2c^2-\frac{1}{2}}{2c+1}$

21. Simplify $\dfrac{\dfrac{1}{a-x}-\dfrac{1}{a-y}+\dfrac{x}{(a-x)^2}-\dfrac{y}{(a-y)^2}}{\dfrac{1}{(a-y)(a-x)^2}-\dfrac{1}{(a-x)(a-y)^2}}$

22. Simplify $\dfrac{1}{x+\dfrac{1}{1+\dfrac{x+1}{3-x}}}$

23. $\dfrac{(x^2-y^2)(2x^2-2xy)}{4(x-y)^2 \div \dfrac{xy}{x+y}}$

24. Simplify $\left(\dfrac{c-b}{c+b}-\dfrac{c^3-b^3}{c^3+b^3}\right)\div\left(\dfrac{c+b}{c-b}+\dfrac{c^2+b^2}{c^2-b^2}\right)$

25. Simplify $\dfrac{y}{(x-y)(x-z)}+\dfrac{x}{(y-x)(y-z)}+\dfrac{x+y}{(z-x)(z-y)}$

26. Simplify $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-a)(b-c)}-\dfrac{1}{abc}$

27. Simplify $\dfrac{x-4+\dfrac{6}{x+1}}{x-\dfrac{6}{x-1}} \times \dfrac{1-\dfrac{x+5}{x^2-1}}{(x-1)(x-2)}$

CHAPTER IX.

FRACTIONAL EQUATIONS.

TO REDUCE EQUATIONS CONTAINING FRACTIONS.

180. (1) $\frac{x}{2}+\frac{x}{4}=12.$

Multiply both sides by 4, the L. C. M. of the denominators.

$$\begin{aligned} \text{Then,}\qquad 2x+x&=48,\\ 3x&=48,\\ \therefore\ x&=16.\end{aligned}$$

(2) $\frac{x}{6}-4=24-\frac{x}{8}.$

Multiply both sides by 24, the L. C. M. of the denominators.

$$\begin{aligned} \text{Then,}\qquad 4x-96&=576-3x,\\ 4x+3x&=576+96,\\ 7x&=672,\\ \therefore\ x&=96.\end{aligned}$$

(3) $\frac{x}{3}-\frac{x-1}{11}=x-9.$

Multiply by 33, the L. C. M. of the denominators.

$$\begin{aligned} \text{Then,}\qquad 11x-3x+3&=33x-297,\\ 11x-3x-33x&=-297-3,\\ -25x&=-300,\\ \therefore\ x&=12.\end{aligned}$$

Since the minus sign precedes the second fraction, in removing the denominator, the + (understood) before x, the first term of the numerator, is changed to −, and the − before 1, the second term of the numerator, is changed to +.

181. Therefore, to clear an equation of fractions,

Multiply each term by the L. C. M. *of the denominators.*

If a fraction is preceded by a **minus sign,** *the sign of every term of the numerator must be changed when the denominator is removed.*

EXERCISE LXIII.

Solve the equations:

1. $5x - \frac{x+2}{2} = 71.$

2. $x - \frac{3-x}{3} = \frac{17}{3}.$

3. $\frac{5-2x}{4} + 2 = x - \frac{6x-8}{2}.$

4. $\frac{5x}{2} - \frac{5x}{4} = \frac{9}{4} - \frac{3-x}{2}.$

5. $2x - \frac{5x-4}{6} = 7 - \frac{1-2x}{5}.$

6. $\frac{x+2}{2} = \frac{14}{9} - \frac{3+5x}{4}.$

7. $\frac{5x+3}{8} - \frac{3-4x}{3} + \frac{x}{2} = \frac{31}{2} - \frac{9-5x}{6}.$

8. $\frac{10x+3}{3} - \frac{6x-7}{2} = 10(x-1).$

9. $\frac{5x-7}{2} - \frac{2x+7}{3} = 3x - 14.$

10. $\frac{7x+5}{6} - \frac{5x-6}{4} = \frac{8-5x}{12}.$

11. $\frac{x+4}{3} - \frac{x-4}{5} = 2 + \frac{3x-1}{15}.$

12. $\frac{3x+5}{7} - \frac{2x+7}{3} + 10 - \frac{3x}{5} = 0.$

13. $\frac{1}{7}(3x-4) + \frac{1}{3}(5x+3) = 43 - 5x.$

14. $\frac{1}{2}(27-2x) = \frac{9}{2} - \frac{1}{10}(7x-54).$

15. $5x-\{8x-3[16-6x-(4-5x)]\}=6.$

16. $\dfrac{5x-3}{7}-\dfrac{9-x}{3}=\dfrac{5x}{2}+\dfrac{19}{6}(x-4).$

17. $\dfrac{2x+7}{7}-\dfrac{9x-8}{11}=\dfrac{x-11}{2}.$

18. $\dfrac{8x-15}{3}-\dfrac{11x-1}{7}=\dfrac{7x+2}{13}.$

19. $\dfrac{7x+9}{8}-\dfrac{3x+1}{7}=\dfrac{9x-13}{4}-\dfrac{249-9x}{14}.$

182. If the denominators contain both simple and compound expressions, it is best to remove the simple expressions first, and then each compound expression in turn. After each multiplication the result should be reduced to the simplest form.

(1) $\dfrac{8x+5}{14}+\dfrac{7x-3}{6x+2}=\dfrac{4x+6}{7}.$

Multiply both sides by 14.

Then, $\quad 8x+5+\dfrac{49x-21}{3x+1}=8x+12.$

Transpose and combine, $\quad \dfrac{49x-21}{3x+1}=7.$

Multiply by $3x+1$, $\quad 49x-21=21x+7,$

$$28x=28,$$

$$\therefore\ x=1.$$

(2) $\dfrac{3-\dfrac{4x}{9}}{4}=\dfrac{1}{4}-\dfrac{\dfrac{7x}{9}-3}{10}.$

Simplify the complex fractions by multiplying both terms of each fraction by 9.

Then, $\quad \dfrac{27-4x}{36}=\dfrac{1}{4}-\dfrac{7x-27}{90}.$

Multiply both sides by 180.

$$135-20x=45-14x+54,$$

$$-6x=-36,$$

$$\therefore\ x=6.$$

Exercise LXIV.

Solve the equations:

1. $\frac{9x+20}{36}=\frac{4(x-3)}{5x-4}+\frac{x}{4}.$

2. $\frac{9(2x-3)}{14}+\frac{11x-1}{3x+1}=\frac{9x+11}{7}.$

3. $\frac{10x+17}{18}-\frac{12x+2}{13x-16}=\frac{5x-4}{9}.$

4. $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}.$

5. $\frac{18x-22}{39-6x}+2x+\frac{1+16x}{24}=4\tfrac{5}{12}-\frac{101-64x}{24}.$

6. $\frac{6-5x}{15}-\frac{7-2x^2}{14(x-1)}=\frac{1+3x}{21}-\frac{10x-11}{30}+\frac{1}{105}.$

7. $\frac{9x+5}{14}+\frac{8x-7}{6x+2}=\frac{36x+15}{56}+\frac{41}{56}.$

8. $\frac{6x+7}{15}-\frac{2x-2}{7x-6}=\frac{2x+1}{5}.$

9. $\frac{6x+1}{15}-\frac{2x-4}{7x-16}=\frac{2x-1}{5}.$

10. $\frac{7x-6}{35}-\frac{x-5}{6x-101}=\frac{x}{5}.$

183. Literal equations are equations in which all the numbers are represented by letters; the numbers regarded as known numbers are usually represented by the *first* letters of the alphabet.

(1) $(a-x)(a+x)=2a^2+2ax-x^2.$

$$\text{Then,}\quad a^2-x^2=2a^2+2ax-x^2,$$
$$-2ax=a^2,$$
$$\therefore x=-\frac{a}{2}.$$

(2) $(x-a)(x-b)-(x-b)(x-c)=2(x-a)(a-c).$

$$(x^2-ax-bx+ab)-(x^2-bx-cx+bc)=2(ax-cx-a^2+ac),$$
$$x^2-ax-bx+ab-x^2+bx+cx-bc=2ax-2cx-2a^2+2ac.$$
$$\text{That is,}\quad -3ax+3cx=-2a^2+2ac-ab+bc,$$
$$-3(a-c)x=-2a(a-c)-b(a-c),$$
$$-3x=-2a-b,$$
$$\therefore x=\frac{2a+b}{3}.$$

Exercise LXV.

Solve the equations:

1. $ax+bc=bx+ac.$

2. $2a-cx=3c-5bx.$

3. $a^2x+bx-c=b^2x+cx-d.$

4. $-ac^2+b^2c+abcx=abc+cmx-ac^2x+b^2c-mc.$

5. $(a+x+b)(a+b-x)=(a+x)(b-x)-ab.$

6. $(a^2+x)^2=x^2+4a^2+a^4.$

7. $(a^2-x)(a^2+x)=a^4+2ax-x^2.$

8. $\dfrac{ax-b}{c}+a=\dfrac{x+ac}{c}.$

9. $\dfrac{a(b^2x+x^3)}{bx}=acx+\dfrac{ax^2}{b}.$

10. $ax-\dfrac{3a-bx}{2}=\dfrac{1}{2}.$

11. $6a-\dfrac{4ax-2b}{3}=x.$

12. $\dfrac{x^2-a}{bx}-\dfrac{a-x}{b}=\dfrac{2x}{b}-\dfrac{a}{x}.$

13. $\dfrac{3}{c}-\dfrac{ab-x^2}{bx}=\dfrac{4x-ac}{cx}.$

14. $am-b-\dfrac{ax}{b}+\dfrac{x}{m}=0.$

15. $\frac{3ax-2b}{3b}-\frac{ax-a}{2b}=\frac{ax}{b}-\frac{2}{3}.$

16. $\frac{ab+x}{b^2}-\frac{b^2-x}{a^2b}=\frac{x-b}{a^2}-\frac{ab-x}{b^2}.$

17. $ax-\frac{bx+1}{x}=\frac{a(x^2-1)}{x}.$

18. $\frac{ax^2}{b-cx}+a+\frac{ax}{c}=0.$

19. $\frac{ab}{x}=bc+d+\frac{1}{x}.$

20. $\frac{a(d^2+x^2)}{dx}=ac+\frac{ax}{d}.$

Exercise LXVI.

Solve the equations:

1. $\frac{x-3}{4(x-1)}=\frac{x-5}{6(x-1)}+\frac{1}{9}.$

2. $x+\frac{x}{x-1}=\frac{(x-2)(x+4)}{x+1}.$

3. $\frac{7}{x-1}=\frac{6x+1}{x+1}-\frac{3(1+2x^2)}{x^2-1}.$

4. $\frac{1}{2(x-3)}-\frac{1}{3(x-2)}=\frac{x-1}{(x-2)(x-3)}.$

5. $1-\frac{2(2x+3)}{9(7-x)}=\frac{6}{7-x}-\frac{5x+1}{4(7-x)}.$

6. $\frac{17}{x+3}-4=\frac{5(21+2x)}{3x+9}-10.$

7. $\frac{x-7}{x+7}=\frac{2x-15}{2x-6}-\frac{1}{2(x+7)}.$

8. $\frac{x+4}{3x+5}+1\frac{1}{6}=\frac{3x+8}{2x+3}.$

9. $\dfrac{132x+1}{3x+1}+\dfrac{8x+5}{x-1}=52.$

10. $\dfrac{2}{2x-3}+\dfrac{1}{x-2}=\dfrac{6}{3x+2}.$

11. $\dfrac{3x-1}{2x-1}-\dfrac{4x-2}{3x-2}=\dfrac{1}{6}.$

12. $\dfrac{3}{x-1}-\dfrac{x+1}{x-1}=\dfrac{x^2}{1-x^2}.$

13. $\dfrac{x-4}{x-5}-\dfrac{x-5}{x-6}=\dfrac{x-7}{x-8}-\dfrac{x-8}{x-9}.$

14. $(x-a)(x-b)=(x-a-b)^2.$

15. $(a-b)(x-c)-(b-c)(x-a)-(c-a)(x-b)=0.$

16. $\dfrac{x^2-x+1}{x-1}+\dfrac{x^2+x+1}{x+1}=2x.$

17. $\dfrac{4}{x+2}+\dfrac{7}{x+3}=\dfrac{37}{x^2+5x+6}.$

18. $(x+1)^2=x[6-(1-x)]-2.$

19. $\dfrac{25-\frac{1}{3}x}{x+1}+\dfrac{16x+4\frac{1}{5}}{3x+2}=\dfrac{23}{x+1}+5.$

20. $\dfrac{3abc}{a+b}+\dfrac{a^2b^2}{(a+b)^3}+\dfrac{(2a+b)\,b^2x}{a(a+b)^2}=3cx+\dfrac{bx}{a}.$

21. $\dfrac{4}{x-8}+\dfrac{3}{2x-16}-\dfrac{29}{24}=\dfrac{2}{3x-24}.$

22. $5-x\left(\dfrac{7}{2}-\dfrac{2}{x}\right)=\dfrac{x}{2}-\dfrac{3x-(4-5x)}{4}.$

23. $\dfrac{1}{5}-\dfrac{3}{x-1}=\dfrac{2+\dfrac{x+4}{1-x}}{3}.$

24. $\dfrac{x-\frac{3}{2}}{\frac{3}{2}(x-1)}+\dfrac{x-\frac{5}{2}}{\frac{5}{2}(x+1)}=1+\dfrac{1}{15\left(1-\dfrac{1}{x^2}\right)}.$

CHAPTER X.

PROBLEMS.

EXERCISE LXVII.

Ex. Find the number the sum of whose third and fourth parts is equal to 12.

Let $x =$ the number.

Then $\frac{x}{3} =$ the third part of the number,

and $\frac{x}{4} =$ the fourth part of the number,

$\therefore \frac{x}{3} + \frac{x}{4} =$ the sum of the two parts.

But $12 =$ the sum of the two parts,

$$\therefore \frac{x}{3} + \frac{x}{4} = 12.$$

Multiply both sides by 12:

$$4x + 3x = 144,$$
$$7x = 144,$$
$$\therefore x = 20\tfrac{4}{7}.$$

1. Find the number whose third and fourth parts together make 14.
2. Find the number whose third part exceeds its fourth part by 14.
3. The half, fourth, and fifth of a certain number are together equal to 76; find the number.
4. Find the number whose double exceeds its half by 12.
5. Divide 60 into two such parts that a seventh of one part may be equal to an eighth of the other.

6. Divide 50 into two such parts that a fourth of one part increased by five-sixths of the other part may be equal to 40.

7. Divide 100 into two such parts that a fourth of one part diminished by a third of the other part may be equal to 11.

8. The sum of the fourth, fifth, and sixth parts of a certain number exceeds the half of the number by 112. What is the number?

9. The sum of two numbers is 5760, and their difference is equal to one-third of the greater. What are the numbers?

10. Divide 45 into two such parts that the first part divided by 2 shall be equal to the second part multiplied by 2.

11. Find a number such that the sum of its fifth and its seventh parts shall exceed the difference of its fourth and its seventh parts by 99.

12. In a mixture of wine and water, the wine was 25 gallons more than half of the mixture, and the water 5 gallons less than one-third of the mixture. How many gallons were there of each?

13. In a certain weight of gunpowder the saltpetre was 6 pounds more than half of the weight, the sulphur 5 pounds less than the third, and the charcoal 3 pounds less than the fourth of the weight. How many pounds were there of each?

14. Divide 46 into two parts such that if one part be divided by 7, and the other by 3, the sum of the quotients shall be 10.

15. A house and garden cost \$850, and five times the price of the house was equal to twelve times the price of the garden. What is the price of each?

16. A man leaves the half of his property to his wife, a sixth to each of his two children, a twelfth to his brother, and the remainder, amounting to \$600, to his sister. What was the amount of his property?

17. The sum of two numbers is a and their difference is b; find the numbers.

18. Find two numbers of which the sum is 70, such that the first divided by the second gives 2 as a quotient and 1 as a remainder.

19. Find two numbers of which the difference is 25, such that the second divided by the first gives 4 as a quotient and 4 as a remainder.

20. Divide the number 208 into two parts such that the sum of the fourth of the greater and the third of the smaller is less by 4 than four times the difference of the two parts.

21. Find four consecutive numbers whose sum is 82.

NOTE I. It is to be remembered that if x represent a person's age at the present time, his age a years ago will be represented by $x - a$, and a years hence by $x + a$.

Ex. In eight years a boy will be three times as old as he was eight years ago. How old is he?

Let $x =$ the number of years of his age.

Then $x - 8 =$ the number of years of his age eight years ago,

and $x + 8 =$ the number of years of his age eight years hence,

$$\begin{aligned}
\therefore\ x + 8 &= 3(x - 8),\\
x + 8 &= 3x - 24,\\
x - 3x &= -24 - 8,\\
-2x &= -32,\\
x &= 16.
\end{aligned}$$

22. A is 72 years old, and B's age is two-thirds of A's. How long is it since A was five times as old as B?

23. A mother is 70 years old, her daughter is half that age. How long is it since the mother was three and one-third times as old as the daughter?

24. A father is three times as old as the son; four years ago the father was four times as old as the son then was. What is the age of each?

25. A is twice as old as B, and seven years ago their united ages amounted to as many years as now represent the age of A. Find the ages of A and B.

26. The sum of the ages of a father and son is half what it will be in 25 years; the difference is one-third what the sum will be in 20 years. What is the age of each?

NOTE II. If A can do a piece of work in x days, the part of the work that he can do in one day will be represented by $\frac{1}{x}$. Thus, if he can do the work in 5 days, in 1 day he can do $\frac{1}{5}$ of the work.

Ex. A can do a piece of work in 5 days, and B can do it in 4 days. How long will it take A and B together to do the work?

Let x = the number of days it will take A and B together.
Then $\frac{1}{x}$ = the part they can do in one day.
Now, $\frac{1}{5}$ = the part A can do in one day,
and $\frac{1}{4}$ = the part B can do in one day.
$\therefore \frac{1}{5} + \frac{1}{4}$ = the part A and B can do in one day.

$$\therefore \tfrac{1}{5} + \tfrac{1}{4} = \tfrac{1}{x},$$
$$4x + 5x = 20,$$
$$9x = 20,$$
$$x = 2\tfrac{2}{9}.$$

Therefore they will do the work in $2\frac{2}{9}$ days.

27. A can do a piece of work in 5 days, B in 6 days, and C in $7\frac{1}{2}$ days; in what time will they do it, all working together?

28. A can do a piece of work in $2\frac{1}{2}$ days, B in $3\frac{1}{3}$ days, and C in $3\frac{3}{4}$ days; in what time will they do it, all working together?

29. Two men who can separately do a piece of work in 15 days and 16 days, can, with the help of another, do it in 6 days. How long would it take the third man to do it alone?

30. A can do half as much work as B, B can do half as much as C, and together they can complete a piece of work in 24 days. In what time can each alone complete the work?

31. A does $\frac{5}{9}$ of a piece of work in 10 days, when B comes to help him, and they finish the work in 3 days more. How long would it have taken B alone to do the whole work?

32. A and B together can reap a field in 12 hours, A and C in 16 hours, and A by himself in 20 hours. In what time can B and C together reap it? In what time can A, B, and C together reap it?

33. A and B together can do a piece of work in 12 days, A and C in 15 days, B and C in 20 days. In what time can they do it, all working together?

NOTE III. If a pipe can fill a vessel in x hours, the part of the vessel filled by it in one hour will be represented by $\frac{1}{x}$. Thus, if a pipe will fill a vessel in 3 hours, in 1 hour it will fill $\frac{1}{3}$ of the vessel.

34. A tank can be filled by two pipes in 24 minutes and 30 minutes respectively, and emptied by a third in 20 minutes. In what time will it be filled if all three are running together?

35. A tank can be filled in 15 minutes by two pipes, A and B, running together. After A has been running by

itself for 5 minutes, B is also turned on, and the tank is filled in 13 minutes more. In what time may it be filled by each pipe separately?

36. A cistern could be filled by two pipes in 6 hours and 8 hours respectively, and could be emptied by a third in 12 hours. In what time would the cistern be filled if the pipes were all running together?

37. A tank can be filled by three pipes in 1 hour and 20 minutes, 3 hours and 20 minutes, and 5 hours, respectively. In what time will the tank be filled when all three pipes are running together?

38. If three pipes can fill a cistern in a, b, and c minutes, respectively, in what time will it be filled by all three running together?

39. The capacity of a cistern is $755\frac{1}{4}$ gallons. The cistern has three pipes, of which the first lets in 12 gallons in $3\frac{1}{4}$ minutes, the second $15\frac{1}{8}$ gallons in $2\frac{1}{2}$ minutes, the third 17 gallons in 3 minutes. In what time will the cistern be filled by the three pipes running together?

NOTE IV. In questions involving distance, time, and rate:

$$\frac{\text{Distance}}{\text{Rate}} = \text{Time}.$$

Thus, if a man travels 40 miles at the rate of 4 miles an hour,

$$\frac{40}{4} = \text{number of hours required.}$$

Ex. A courier who goes at the rate of $31\frac{1}{2}$ miles in 5 hours, is followed, after 8 hours, by another who goes at the rate of $22\frac{1}{2}$ miles in 3 hours. In how many hours will the second overtake the first?

Since the first goes $31\frac{1}{2}$ miles in 5 hours, his rate per hour is $6\frac{3}{10}$ miles.

Since the second goes $22\frac{1}{2}$ miles in 3 hours, his rate per hour is $7\frac{1}{2}$ miles.

Let $x =$ the number of hours the first is travelling.

Then $x - 8 =$ the number of hours the second is travelling.

Then $6\frac{3}{10}\,x =$ the number of miles the first travels;

$(x-8)\,7\frac{1}{2} =$ the number of miles the second travels.

They both travel the same distance,

$$\therefore\ 6\tfrac{3}{10}\,x = (x-8)\,7\tfrac{1}{2}.$$

The solution of which gives 42 hours.

40. A sets out and travels at the rate of 7 miles in 5 hours. Eight hours afterwards, B sets out from the same place and travels in the same direction, at the rate of 5 miles in 3 hours. In how many hours will B overtake A?

41. A person walks to the top of a mountain at the rate of $2\frac{1}{3}$ miles an hour, and down the same way at the rate of $3\frac{1}{2}$ miles an hour, and is out 5 hours. How far is it to the top of the mountain?

42. A person has a hours at his disposal. How far may he ride in a coach which travels b miles an hour, so as to return home in time, walking back at the rate of c miles an hour?

43. The distance between London and Edinburgh is 360 miles. One traveller starts from Edinburgh and travels at the rate of 10 miles an hour; another starts at the same time from London, and travels at the rate of 8 miles an hour. How far from London will they meet?

44. Two persons set out from the same place in opposite directions. The rate of one of them per hour is a mile less than double that of the other, and in 4 hours they are 32 miles apart. Determine their rates.

45. In going a certain distance, a train travelling 35 miles an hour takes 2 hours less than one travelling 25 miles an hour. Determine the distance.

NOTE V. In problems relating to clocks, it is to be observed that the minute-hand moves *twelve times* as fast as the hour-hand.

Ex. Find the time between two and three o'clock when the hands of a clock are:

I. Together.
II. At right angles to each other.
III. Opposite to each other.

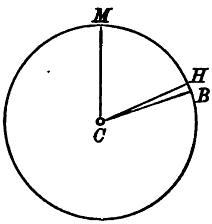


Fig. 1.

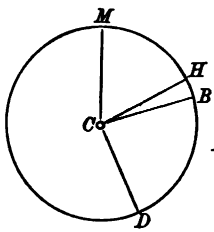


Fig. 2.

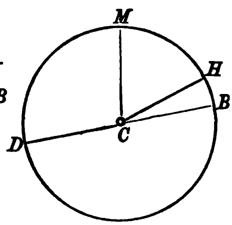


Fig. 3.

I. Let CH and CM (Fig. 1) denote the positions of the hour and minute hands at 2 o'clock, and CB the position of both hands when together.

Then arc HB = *one-twelfth* of arc MB.

Let x = number of minute-spaces in arc MB.

Then $\frac{x}{12}$ = number of minute-spaces in arc HB,

and 10 = number of minute-spaces in arc MH.

Now arc MB = arc MH + arc HB.

That is, $$x = 10 + \frac{x}{12}.$$

The solution of this equation gives $x = 10\frac{10}{11}$.

Hence, the time is $10\frac{10}{11}$ minutes past 2 o'clock.

II. Let CB and CD (Fig. 2) denote the positions of the hour and minute hands when at right angles to each other.

Let $\quad x =$ number of minute-spaces in arc $MHBD$.

Then $\quad \frac{x}{12} =$ number of minute-spaces in arc HB,

and $\quad 10 =$ number of minute-spaces in arc MH.

$15 =$ number of minute-spaces in arc BD.

Now arc $MHBD =$ arcs $MH + HB + BD$.

That is, $\quad x = 10 + \frac{x}{12} + 15.$

The solution of this equation gives $x = 27\frac{3}{11}$.

Hence, the time is $27\frac{3}{11}$ minutes past 2 o'clock.

III. Let CB and CD (Fig. 3) denote the positions of the hour and minute hands when opposite to each other.

Let $\quad x =$ number of minute-spaces in arc $MHBD$.

Then $\quad \frac{x}{12} =$ number of minute-spaces in arc HB,

and $\quad 10 =$ number of minute-spaces in arc MH,

$30 =$ number of minute-spaces in arc BD.

Now arc $MHBD =$ arcs $MH + HB + BD$.

That is, $\quad x = 10 + \frac{x}{12} + 30.$

The solution of this equation gives $x = 43\frac{7}{11}$.

Hence, the time is $43\frac{7}{11}$ minutes past 2 o'clock.

46. At what time are the hands of a watch together:

I. Between 3 and 4?
II. Between 6 and 7?
III. Between 9 and 10?

47. At what time are the hands of a watch at right angles:

I. Between 3 and 4?
II. Between 4 and 5?
III. Between 7 and 8?

48. At what time are the hands of a watch opposite to each other:

I. Between 1 and 2?
II. Between 4 and 5?
III. Between 8 and 9?

49. It is between 2 and 3 o'clock; but a person looking at his watch and mistaking the hour-hand for the minute hand, fancies that the time of day is 55 minutes earlier than it really is. What is the true time?

NOTE VI. It is to be observed that if a represent the number of feet in the length of a step or leap, and x the number of steps or leaps taken, then ax will represent the number of feet in the distance made.

Ex. A hare takes 4 leaps to a greyhound's 3; but 2 of the greyhound's leaps are equivalent to 3 of the hare's. The hare has a start of 50 leaps. How many leaps must the greyhound take to catch the hare?

Let $3x =$ the number of leaps taken by the greyhound.
Then $4x =$ the number of leaps of the hare in the same time.
Also, let a denote the number of feet in one leap of the hare.
Then $\frac{3a}{2}$ will denote the number of feet in one leap of the greyhound.

That is, $$3x \times \frac{3a}{2} = \text{the whole distance},$$

and $$(50 + 4x)\,a = \text{the whole distance},$$

$$\therefore \frac{9ax}{2} = (50 + 4x)\,a.$$

Divide by a, $$\frac{9x}{2} = 50 + 4x,$$

$$9x = 100 + 8x,$$

$$x = 100,$$

$$\therefore 3x = 300.$$

Thus the greyhound must take 300 leaps.

50. A hare takes 6 leaps to a dog's 5, and 7 of the dog's leaps are equivalent to 9 of the hare's. The hare has a start of 50 of her own leaps. How many leaps will the hare take before she is caught?

51. A greyhound makes 3 leaps while a hare makes 4; but 2 of the greyhound's leaps are equivalent to 3 of the hare's. The hare has a start of 50 of the greyhound's leaps. How many leaps does each take before the hare is caught?

52. A greyhound makes two leaps while a hare makes 3; but 1 leap of the greyhound is equivalent to 2 of the hare's. The hare has a start of 80 of her own leaps. How many leaps will the hare take before she is caught?

NOTE VII. It is to be observed that if the number of units in the breadth and length of a rectangle be represented by x and $x + a$, respectively, then $x(x + a)$ will represent the number of surface units in the rectangle, the unit of surface having the same name as the linear unit in which the sides of the rectangle are expressed.

53. A rectangle whose length is 5 feet more than its breadth would have its area increased by 22 feet if its length and breadth were each made a foot more. Find its dimensions.

54. A rectangle has its length and breadth respectively 5 feet longer and 3 feet shorter than the side of the equivalent square. Find its area.

55. The length of a rectangle is an inch less than double its breadth; and when a strip 3 inches wide is cut off all round, the area is diminished by 210 inches. Find the size of the rectangle at first.

56. The length of a floor exceeds the breadth by 4 feet; if each dimension were increased by 1 foot, the area of the room would be increased by 27 square feet. Find its dimensions.

NOTE VIII. It is to be observed that if b pounds of metal lose a pounds when weighed in water, 1 pound will lose $\frac{1}{b}$ of a pounds, or $\frac{a}{b}$ of a pound.

57. A mass of tin and lead weighing 180 pounds loses 21 pounds when weighed in water; and it is known that 37 pounds of tin lose 5 pounds, and 23 pounds of lead lose 2 pounds, when weighed in water. How many pounds of tin and of lead in the mass?

58. If 19 pounds of gold lose 1 pound, and 10 pounds of silver lose 1 pound, when weighed in water, find the amount of each in a mass of gold and silver weighing 106 pounds in air and 99 pounds in water.

59. Fifteen sovereigns should weigh 77 pennyweights; but a parcel of light sovereigns, having been weighed and counted, was found to contain 9 more than was supposed from the weight; and it appeared that 21 of these coins weighed the same as 20 true sovereigns. How many were there altogether?

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60. There are two silver cups, and one cover for both. The first weighs 12 ounces, and with the cover weighs twice as much as the other without it; but the second with the cover weighs one-third more than the first without it. Find the weight of the cover.

61. A man wishes to enclose a circular piece of ground with palisades, and finds that if he sets them a foot apart he will have too few by 150; but if he sets them a yard apart he will have too many by 70. What is the circuit of the piece of ground?

62. A horse was sold at a loss for $200; but if it had been sold for $250, the gain would have been three-fourths of the loss when sold for $200. Find the value of the horse.

63. A and B shoot by turns at a target. A puts 7 bullets out of 12, and B 9 out of 12, into the centre. Between them they put in 32 bullets. How many shots did each fire?

64. A boy buys a number of apples at the rate of 5 for 2 pence. He sells half of them at 2 a penny and the rest at 3 a penny, and clears a penny by the transaction. How many does he buy?

65. A person bought a piece of land for $6750, of which he kept $\frac{1}{3}$ for himself. At the cost of $250 he made a road which took $\frac{1}{10}$ of the remainder, and then sold the rest at $12\frac{1}{2}$ cents a square yard more than double the price it cost him, thus clearing his outlay and $500 besides. How much land did he buy, and what was the cost-price per yard?

66. A boy who runs at the rate of 12 yards per second starts 20 yards behind another whose rate is $10\frac{1}{2}$ yards per second. How soon will the first boy be 10 yards ahead of the second?

67. A merchant adds yearly to his capital one-third of it, but takes from it, at the end of each year, $5000 for expenses. At the end of the third year, after deducting the last $5000, he has twice his original capital. How much had he at first?

68. A shepherd lost a number of sheep equal to one-fourth of his flock and one-fourth of a sheep; then, he lost a number equal to one-third of what he had left and one-third of a sheep; finally, he lost a number equal to one-half of what now remained and one-half a sheep, after which he had but 25 sheep left. How many had he at first?

69. A trader maintained himself for three years at an expense of $250 a year; and each year increased that part of his stock which was not so expended by one-third of it. At the end of the third year his original stock was doubled. What was his original stock?

70. A cask contains 12 gallons of wine and 18 gallons of water; another cask contains 9 gallons of wine and 3 gallons of water. How many gallons must be drawn from each cask to produce a mixture containing 7 gallons of wine and 7 gallons of water?

71. The members of a club subscribe each as many dollars as there are members. If there had been 12 more members, the subscription from each would have been \$10 less, to amount to the same sum. How many members were there?

72. A number of troops being formed into a solid square, it was found there were 60 men over; but when formed in a column with 5 men more in front than before, and 3 men less in depth, there was lacking one man to complete it. Find the number of troops.

73. An officer can form the men of his regiment into a hollow square twelve deep. The number of men in the regiment is 1296. Find the number of men in the front of the hollow square.

74. A person starts from P and walks towards Q at the rate of 3 miles an hour; 20 minutes later another person starts from Q and walks towards P at the rate of 4 miles an hour. The distance from P to Q is 20 miles. How far from P will they meet?

75. A person engaged to work a days on these conditions: for each day he worked he was to receive b cents, and for each day he was idle he was to forfeit c cents. At the end of a days he received d cents. How many days was he idle?

76. A banker has two kinds of coins: it takes a pieces of the first to make a dollar, and b pieces of the second to make a dollar. A person wishes to obtain c pieces for a dollar. How many pieces of each kind must the banker give him?

CHAPTER XI.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

184. If *one* equation contain *two* unknown quantities, an *indefinite number of pairs* of values may be found that will satisfy the equation.

Thus, in the equation $x+y=10$, *any values* may be given to x, and *corresponding values* for y may be found. *Any pair* of these values substituted for x and y will satisfy the equation.

185. But if a second equation be given, expressing *different relations* between the unknown quantities, only *one pair* of values of x and y can be found that will satisfy *both* equations.

Thus, if besides the equation $x+y=10$, another equation, $x-y=2$, be given, it is evident that the values of x and y which will satisfy both equations are

$$\left.\begin{array}{l} x=6 \\ y=4 \end{array}\right\},$$

for $6+4=10$, and $6-4=2$; and these are the *only* values of x and y that will satisfy *both* equations.

186. Equations that express *different* relations between the unknown quantities are called **independent equations.**

Thus, $x+y=10$ and $x-y=2$ are independent equations; they express *different* relations between x and y. But $x+y=10$ and $3x+3y=30$ are not independent

equations; one is derived immediately from the other, and both express the *same* relation between the unknown quantities.

187. Equations that are to be satisfied by the *same values* of the unknown quantities are called **simultaneous equations.**

188. Simultaneous equations are solved by combining the equations so as to obtain a single equation containing only one unknown quantity; and this process is called **elimination.**

Three methods of elimination are generally given:

I. By Addition or Subtraction.
II. By Substitution.
III. By Comparison.

Elimination by Addition or Subtraction.

(1) Solve:
$$\left.\begin{aligned} 2x-3y&=4 \\ 3x+2y&=32 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (1) by 2 and (2) by 3,

$$4x-6y=8 \qquad (3)$$
$$9x+6y=96 \qquad (4)$$

Add (3) and (4), $13x=104$

$$\therefore x=8.$$

Substitute the value of x in (2),

$$24+2y=32,$$
$$\therefore y=4.$$

In this solution y is eliminated by *addition.*

(2) Solve:
$$\left.\begin{aligned} 6x+35y&=177 \\ 8x-21y&=33 \end{aligned}\right\} \qquad \begin{aligned}&(1)\\&(2)\end{aligned}$$

Multiply (1) by 4 and (2) by 3,

$$24x+140y=708 \qquad (3)$$
$$24x-63y=99 \qquad (4)$$

Subtract (4) from (3), $203y=609$

$$\therefore y=3.$$

Substitute the value of y in (2).

$$8x - 63 = 33.$$

$$\therefore\ x = 12.$$

In this solution x is eliminated by *subtraction*.

189. Hence, to eliminate an unknown quantity by addition or subtraction,

Multiply the equations by such numbers as will make the coefficients of this unknown quantity equal in the resulting equations.

Add the resulting equations, or subtract one from the other, according as these equal quantities have unlike or like signs.

NOTE. It is generally best to select that unknown quantity to be eliminated which requires the smallest multipliers to make its coefficients equal; and the smallest multiplier for each equation is found by dividing the L. C. M. of the coefficients of this unknown quantity by the given coefficient in that equation. Thus, in example (2), the L. C. M. of 6 and 8 (the coefficients of x), is 24, and hence the smallest multipliers of the two equations are 4 and 3 respectively.

Sometimes the solution is simplified by first adding the given equations, or by subtracting one from the other.

(3)

$$x + 49y = 51 \qquad (1)$$

$$49x + y = 99 \qquad (2)$$

Add (1) and (2), $\quad 50x + 50y = 150 \qquad (3)$

Divide (3) by 50, $\quad x + y = 3. \qquad (4)$

Subtract (4) from (1), $\quad 48y = 48,$

$$\therefore\ y = 1.$$

Subtract (4) from (2), $\quad 48x = 96,$

$$\therefore\ x = 2.$$

EXERCISE LXVIII.

Solve by addition or subtraction:

1. $\left.\begin{array}{l} 2x + 3y = 7 \\ 4x - 5y = 3 \end{array}\right\}$

2. $\left.\begin{array}{l} x - 2y = 4 \\ 2x - y = 5 \end{array}\right\}$

3. $\left.\begin{array}{l} 7x + 2y = 30 \\ y - 3x = 2 \end{array}\right\}$

4. $\left.\begin{array}{l} 3x - 5y = 51 \\ 2x + 7y = 3 \end{array}\right\}$

5. $\left.\begin{array}{l} 5x + 4y = 58 \\ 3x + 7y = 67 \end{array}\right\}$

6. $\left.\begin{array}{l} 3x + 2y = 39 \\ 3y - 2x = 13 \end{array}\right\}$

7. $\left.\begin{aligned} 3x - 4y &= -5 \\ 4x - 5y &= 1 \end{aligned}\right\}$

8. $\left.\begin{aligned} 11x + 3y &= 100 \\ 4x - 7y &= 4 \end{aligned}\right\}$

9. $\left.\begin{aligned} x + 49y &= 693 \\ 49x + y &= 357 \end{aligned}\right\}$

10. $\left.\begin{aligned} 17x + 3y &= 57 \\ 16y - 3x &= 23 \end{aligned}\right\}$

11. $\left.\begin{aligned} 12x + 7y &= 176 \\ 3y - 19x &= 3 \end{aligned}\right\}$

12. $\left.\begin{aligned} 2x - 7y &= 8 \\ 4y - 9x &= 19 \end{aligned}\right\}$

13. $\left.\begin{aligned} 69y - 17x &= 103 \\ 14x - 13y &= -41 \end{aligned}\right\}$

14. $\left.\begin{aligned} 17x + 30y &= 59 \\ 19x + 28y &= 77 \end{aligned}\right\}$

Elimination by Substitution.

(1) Solve: $\left.\begin{aligned} 2x + 3y &= 8 \\ 3x + 7y &= 7 \end{aligned}\right\}$

$$2x + 3y = 8 \qquad (1)$$
$$3x + 7y = 7 \qquad (2)$$

Transpose $3y$ in (1), $2x = 8 - 3y$.

Divide by coefficient of x, $x = \dfrac{8 - 3y}{2}$. (4)

Substitute the value of x in (2), $3\left(\dfrac{8 - 3y}{2}\right) + 7y = 7,$

$$\frac{24 - 9y}{2} + 7y = 7,$$
$$24 - 9y + 14y = 14,$$
$$5y = -10,$$
$$\therefore y = -2.$$

Substitute the value of y in (1), $2x - 6 = 8,$

$$\therefore x = 7.$$

190. Hence, to eliminate an unknown quantity by substitution,

From one of the equations obtain the value of one of the unknown quantities in terms of the other.

Substitute for this unknown quantity its value in the other equation, and reduce the resulting equation.

Exercise LXIX.

Solve by substitution:

1. $\left.\begin{array}{l}3x-4y=2\\7x-9y=7\end{array}\right\}$

2. $\left.\begin{array}{l}7x-5y=24\\4x-3y=11\end{array}\right\}$

3. $\left.\begin{array}{r}3x+2y=32\\20x-3y=\ \ 1\end{array}\right\}$.

4. $\left.\begin{array}{r}11x-7y=37\\8x+9y=41\end{array}\right\}$

5. $\left.\begin{array}{r}7x+\ \ 5y=60\\13x-11y=10\end{array}\right\}$

6. $\left.\begin{array}{l}6x-7y=42\\7x-6y=75\end{array}\right\}$

7. $\left.\begin{array}{l}10x+\ \ 9y=290\\12x-11y=130\end{array}\right\}$

8. $\left.\begin{array}{l}3x-4y=18\\3x+2y=\ \ 0\end{array}\right\}$

9. $\left.\begin{array}{l}9x-5y=52\\8y-3x=\ \ 8\end{array}\right\}$

10. $\left.\begin{array}{r}5x-3y=\ \ 4\\12y-7x=10\end{array}\right\}$

11. $\left.\begin{array}{r}9y-7x=13\\15x-7y=\ \ 9\end{array}\right\}$

12. $\left.\begin{array}{r}5x-2y=\ \ 51\\19x-3y=180\end{array}\right\}$

13. $\left.\begin{array}{l}4x+\ \ 9y=106\\8x+17y=198\end{array}\right\}$

14. $\left.\begin{array}{r}8x+3y=3\\12x+9y=3\end{array}\right\}$

Elimination by Comparison.

Solve: $\left.\begin{array}{l}2x-9y=11\\3x-4y=7\end{array}\right\}$

$$2x-9y=11, \qquad (1)$$
$$3x-4y=7. \qquad (2)$$

Transpose $9y$ in (1) and $4y$ in (2), $2x=11+9y$, (3)

$$3x=7+4y. \qquad (4)$$

Divide (3) by 2 and (4) by 3, $x=\frac{11+9y}{2}$, (5)

$$x=\frac{7+4y}{3}. \qquad (6)$$

Equate the values of x, $\frac{11+9y}{2}=\frac{7+4y}{3}$. (7)

Reduce (7) $\quad 33 + 27y = 14 + 8y,$

$$19y = -19,$$

$$\therefore\ y = -1.$$

Substitute the value of y in (1), $2x + 9 = 11,$

$$\therefore\ x = 1.$$

191. Hence, to eliminate an unknown quantity by comparison,

From each equation obtain the value of one of the unknown quantities in terms of the other.

Form an equation from these equal values and reduce the equation.

NOTE. If, in the last example, (3) be divided by (4), the resulting equation, $\frac{2}{3} = \frac{11 + 9y}{7 + 4y}$, would, when reduced, give the value of y. This is the shortest method, and therefore to be preferred.

EXERCISE LXX.

Solve by comparison:

1. $\left.\begin{aligned} x + 15y &= 53 \\ 3x + y &= 27 \end{aligned}\right\}$

2. $\left.\begin{aligned} 4x + 9y &= 51 \\ 8x - 13y &= 9 \end{aligned}\right\}$

3. $\left.\begin{aligned} 4x + 3y &= 48 \\ 5y - 3x &= 22 \end{aligned}\right\}$

4. $\left.\begin{aligned} 2x + 3y &= 43 \\ 10x - y &= 7 \end{aligned}\right\}$

5. $\left.\begin{aligned} 5x - 7y &= 33 \\ 11x + 12y &= 100 \end{aligned}\right\}$

6. $\left.\begin{aligned} 5x + 7y &= 43 \\ 11x + 9y &= 69 \end{aligned}\right\}$

7. $\left.\begin{aligned} 8x - 21y &= 33 \\ 6x + 35y &= 177 \end{aligned}\right\}$

8. $\left.\begin{aligned} 3y - 7x &= 4 \\ 2y + 5x &= 22 \end{aligned}\right\}$

9. $\left.\begin{aligned} 21y + 20x &= 165 \\ 77y - 30x &= 295 \end{aligned}\right\}$

10. $\left.\begin{aligned} 11x - 10y &= 14 \\ 5x + 7y &= 41 \end{aligned}\right\}$

11. $\left.\begin{aligned} 7y - 3x &= 139 \\ 2x + 5y &= 91 \end{aligned}\right\}$

12. $\left.\begin{aligned} 17x + 12y &= 59 \\ 19x - 4y &= 153 \end{aligned}\right\}$

13. $\left.\begin{aligned} 24x + 7y &= 27 \\ 8x - 33y &= 115 \end{aligned}\right\}$

14. $\left.\begin{aligned} x &= 3y - 19 \\ y &= 3x - 23 \end{aligned}\right\}$

192. Each equation must be simplified, if necessary, before the elimination is performed.

(1) Solve: $\left.\begin{aligned}(x-1)(y+2)&=(x-3)(y-1)+8\\ \frac{2x-1}{5}-\frac{3(y-2)}{4}&=1\end{aligned}\right\}$

$$(x-1)(y+2)=(x-3)(y-1)+8 \qquad (1)$$
$$\frac{2x-1}{5}-\frac{3(y-2)}{4}=1 \qquad (2)$$

Simplify (1), $xy+2x-y-2=xy-x-3y+3+8.$

Transpose and combine, $3x+2y=13.$ (3)

Simplify (2), $8x-4-15y+30=20.$

Transpose and combine, $8x-15y=-6.$ (4)

Multiply (3) by 8, $24x+16y=104.$ (5)

Multiply (4) by 3, $24x-45y=-18.$ (6)

Subtract (6) from (5), $61y=122,$

$\therefore y=2.$

Substitute the value of y in (3), $3x+4=13,$

$\therefore x=3.$

Exercise LXXI.

Solve:

1. $\left.\begin{aligned}x(y+7)&=y(x+1)\\ 2x+20&=3y+1\end{aligned}\right\}$

2. $\left.\begin{aligned}2x-\frac{y-3}{5}-4&=0\\ 3y+\frac{x-2}{3}-9&=0\end{aligned}\right\}$

3. $\left.\begin{aligned}\frac{2}{x+3}&=\frac{3}{y-2}\\ 5(x+3)&=3(y-2)+2\end{aligned}\right\}$

4. $\left.\begin{aligned}\frac{x-4}{5}-\frac{y+2}{10}&=0\\ \frac{x}{6}+\frac{y-2}{4}&=3\end{aligned}\right\}$

5. $\left.\begin{aligned}(x+1)(y+2)-(x+2)(y+1)&=-1\\ 3(x+3)-4(y+4)&=-8\end{aligned}\right\}$

6. $\left.\begin{aligned}\frac{x-2}{5}-\frac{10-x}{3}&=\frac{y-10}{4}\\ \frac{2y+4}{3}-\frac{2x+y}{8}&=\frac{x+13}{4}\end{aligned}\right\}$

7. $\left.\begin{aligned} \frac{x+1}{3}-\frac{y+2}{4}&=\frac{2(x-y)}{5} \\ \frac{x-3}{4}-\frac{y-3}{3}&=2y-x \end{aligned}\right\}$

8. $\left.\begin{aligned} \frac{3x-2y}{5}+\frac{5x-3y}{3}&=x+1 \\ \frac{2x-3y}{3}+\frac{4x-3y}{2}&=y+1 \end{aligned}\right\}$

9. $\left.\begin{aligned} \frac{2x-y+3}{3}-\frac{x-2y+3}{4}&=4 \\ \frac{3x-4y+3}{4}+\frac{4x-2y-9}{3}&=4 \end{aligned}\right\}$

10. $\left.\begin{aligned} 1\tfrac{1}{2}x&=1\tfrac{1}{3}y+4\tfrac{5}{12} \\ 4\tfrac{1}{2}x&=\tfrac{1}{3}y-21\tfrac{7}{12} \end{aligned}\right\}$

11. $\left.\begin{aligned} \frac{13}{x+2y+3}&=-\frac{3}{4x-5y+6} \\ \frac{3}{6x-5y+4}&=\frac{19}{3x+2y+1} \end{aligned}\right\}$

12. $\left.\begin{aligned} \frac{x+y}{y-x}&=\frac{15}{8} \\ 9x-\frac{3y+44}{7}&=100 \end{aligned}\right\}$

13. $\left.\begin{aligned} \frac{3x-5y}{2}+3&=\frac{2x+y}{5} \\ 8-\frac{x-2y}{4}&=\frac{x}{2}+\frac{y}{3} \end{aligned}\right\}$

14. $\left.\begin{aligned} \frac{4x-3y-7}{5}&=\frac{3x}{10}-\frac{2y}{15}-\frac{5}{6} \\ \frac{y-1}{3}+\frac{x}{2}-\frac{3y}{20}-1&=\frac{y-x}{15}+\frac{x}{6}+\frac{1}{10} \end{aligned}\right\}$

15. $\left.\begin{aligned} \frac{x-4}{5}&=\frac{y+2}{10} \\ \frac{x}{6}+\frac{y-2}{4}&=3 \end{aligned}\right\}$

16. $\left.\begin{aligned} \frac{3x+12y}{11}&=9 \\ \frac{1-3x}{7}&=\frac{11-3y}{5} \end{aligned}\right\}$

17. $\left.\begin{aligned} 5x-\tfrac{1}{4}(5y+2)&=32 \\ 3y+\tfrac{1}{3}(x+2)&=9 \end{aligned}\right\}$

18. $\left.\begin{aligned} 3x-.25y&=28 \\ .12x+.7y&=2.54 \end{aligned}\right\}$

19. $\left.\begin{aligned} 7(x-1)&=3(y+8) \\ \frac{4x+2}{9}&=\frac{5y+9}{2} \end{aligned}\right\}$

20. $\left.\begin{aligned} 7x+\tfrac{1}{5}(2y+4)&=16 \\ 3y-\tfrac{1}{4}(x+2)&=8 \end{aligned}\right\}$

21. $$\left.\begin{aligned} \frac{5x-6y}{13}+3x&=4y-2 \\ \frac{5x+6y}{6}-\frac{3x-2y}{4}&=2y-2 \end{aligned}\right\}$$

22. $$\left.\begin{aligned} \frac{5x-3}{2}-\frac{3x-19}{2}&=4-\frac{3y-x}{3} \\ \frac{2x+y}{2}-\frac{9x-7}{8}&=\frac{3(y+3)}{4}-\frac{4x+5y}{16} \end{aligned}\right\}$$

23. $$\left.\begin{aligned} 3y+11&=\frac{4x^2-y(x+3y)}{x-y+4}+31-4x \\ (x+7)(y-2)+3&=2xy-(y-1)(x+1) \end{aligned}\right\}$$

24. $$\left.\begin{aligned} \frac{6x+9}{4}+\frac{3x+5y}{4x-6}&=3\tfrac{1}{4}+\frac{3x+4}{2} \\ \frac{8y+7}{10}+\frac{6x-3y}{2y-8}&=4+\frac{4y-9}{5} \end{aligned}\right\}$$

25. $$\left.\begin{aligned} x-\frac{2y-x}{23-x}&=20-\frac{59-2x}{2} \\ y+\frac{y-3}{x-18}&=30-\frac{73-3y}{3} \end{aligned}\right\}$$

LITERAL SIMULTANEOUS EQUATIONS.

193. The method of solving literal simultaneous equations is as follows:

Solve:
$$\left.\begin{aligned} ax+by&=m \\ cx+dy&=n \end{aligned}\right\}$$

$$ax+by=m \qquad (1)$$
$$cx+dy=n \qquad (2)$$

Multiply (1) by c, $\quad acx+bcy=cm \qquad (3)$

Multiply (2) by a, $\quad acx+ady=an \qquad (4)$

Subtract (4) from (3), $\quad (bc-ad)\,y=cm-an$

Divide by coefficient of y, $\quad y=\dfrac{cm-an}{bc-ad}$

To find the value of x:

Multiply (1) by d,	$adx + bdy = dm$	(5)
Multiply (2) by b,	$bcx + bdy = bn$	(6)
Subtract (6) from (5),	$(ad - bc)\,x = dm - bn$	
Divide by coefficient of x,	$x = \dfrac{dm - bn}{ad - bc}$	

Exercise LXXII.

Solve:

1. $\left.\begin{array}{l} x + y = a \\ x - y = b \end{array}\right\}$

2. $\left.\begin{array}{l} ax + by = c \\ px + qy = r \end{array}\right\}$

3. $\left.\begin{array}{l} mx + ny = a \\ px + qy = b \end{array}\right\}$

4. $\left.\begin{array}{l} ax + by = e \\ ax + cy = d \end{array}\right\}$

5. $\left.\begin{array}{l} mx - ny = r \\ m'x + n'y = r' \end{array}\right\}$

6. $\left.\begin{array}{l} ax + by = c \\ dx + fy = c^2 \end{array}\right\}$

7. $\left.\begin{array}{l} \dfrac{x}{a} + \dfrac{y}{b} = c \\ \dfrac{x}{b} \div \dfrac{y}{a} = -c \end{array}\right\}$

8. $\left.\begin{array}{l} abx + cdy = 2 \\ ax - cy = \dfrac{d - b}{bd} \end{array}\right\}$

9. $\left.\begin{array}{l} \dfrac{a}{b + y} = \dfrac{b}{3a + x} \\ ax + 2by = d \end{array}\right\}$

10. $\left.\begin{array}{l} \dfrac{x}{a + b} - \dfrac{y}{a - b} = \dfrac{1}{a + b} \\ \dfrac{x}{a + b} + \dfrac{y}{a - b} = \dfrac{1}{a - b} \end{array}\right\}$

11. $\left.\begin{array}{l} a(a - x) = b(x + y - a) \\ a(y - b - x) = b(y - b) \end{array}\right\}$

12. $\left.\begin{array}{l} \dfrac{x - y + 1}{x - y - 1} = a \\ \dfrac{x + y + 1}{x + y - 1} = b \end{array}\right\}$

13. $\left.\begin{array}{l} ax = by + \dfrac{a^2 + b^2}{2} \\ (a - b)\,x = (a + b)\,y \end{array}\right\}$

14. $\left.\begin{array}{l} ax + by = c^2 \\ \dfrac{a}{b + y} - \dfrac{b}{a + x} = 0 \end{array}\right\}$

15. $\left.\begin{array}{l} \dfrac{x}{a + b} + \dfrac{y}{a - b} = 2a \\ \dfrac{x - y}{2ab} = \dfrac{x + y}{a^2 + b^2} \end{array}\right\}$

16. $\left.\begin{array}{l} bx - bc = ay - ac \\ x - y = a - b \end{array}\right\}$

7. $\left.\begin{aligned}\frac{2}{ax}+\frac{3}{by}&=5\\ \frac{5}{ax}-\frac{2}{by}&=3\end{aligned}\right\}$ 8. $\left.\begin{aligned}\frac{m}{nx}+\frac{n}{my}&=m+n\\ \frac{n}{x}+\frac{m}{y}&=m^2+n^2\end{aligned}\right\}$ 9. $\left.\begin{aligned}\frac{a}{x}+\frac{b}{y}&=m\\ \frac{b}{x}-\frac{a}{y}&=n\end{aligned}\right\}$

195. If three simultaneous equations are given, involving three unknown quantities, one of the unknown quantities must be eliminated between *two pairs* of the equations; then a second between the resulting equations.

196. Likewise, if four or more equations are given, involving four or more unknown quantities, one of the unknown quantities must be eliminated between three or more pairs of the equations; then a second between the pairs that can be found of the resulting equations; and so on.

Solve:

$$\left.\begin{aligned}2x-3y+4z&=4\\ 3x+5y-7z&=12\\ 5x-y-8z&=5\end{aligned}\right\}\qquad\begin{aligned}&(1)\\&(2)\\&(3)\end{aligned}$$

Eliminate z between two pairs of these equations.

Multiply (1) by 2,	$4x-6y+8z=8$	(4)
(3) is	$5x-y-8z=5$	
Add,	$9x-7y=13$	(5)
Multiply (1) by 7,	$14x-21y+28z=28$	
Multiply (2) by 4,	$12x+20y-28z=48$	
Add,	$26x-y=76$	(6)
Multiply (6) by 7,	$182x-7y=532$	(7)
(5) is	$9x-7y=13$	
Subtract (5) from (7),	$173x=519$	
	$\therefore x=3.$	

Substitute the value of x in (6), $78-y=76,$

$\therefore y=2.$

Substitute the values of x and y in (1), $6-6+4z=4,$

$\therefore z=1.$

EXERCISE LXXIV.

Solve:

1. $\left.\begin{array}{l}5x+3y-6z=4\\3x-y+2z=8\\x-2y+2z=2\end{array}\right\}$

2. $\left.\begin{array}{l}4x-5y+2z=6\\2x+3y-z=20\\7x-4y+3z=35\end{array}\right\}$

3. $\left.\begin{array}{l}x+y+z=6\\5x+4y+3z=22\\15x+10y+6z=53\end{array}\right\}$

4. $\left.\begin{array}{l}4x-3y+z=9\\9x+y-5z=16\\x-4y+3z=2\end{array}\right\}$

5. $\left.\begin{array}{l}8x+4y-3z=6\\x+3y-z=7\\4x-5y+4z=8\end{array}\right\}$

6. $\left.\begin{array}{l}12x+5y-4z=29\\13x-2y+5z=58\\17x-y-z=15\end{array}\right\}$

7. $\left.\begin{array}{l}y-x+z=-5\\z-y-x=-25\\x+y+z=35\end{array}\right\}$

8. $\left.\begin{array}{l}x+y+z=30\\8x+4y+2z=50\\27x+9y+3z=64\end{array}\right\}$

9. $\left.\begin{array}{l}15y=24z-10x+41\\15x=12y-16z+10\\18x-(7z-13)=14y\end{array}\right\}$

10. $\left.\begin{array}{l}3x-y+z=17\\5x+3y-2z=10\\7x+4y-5z=3\end{array}\right\}$

11. $\left.\begin{array}{l}x+y+z=5\\3x-5y+7z=75\\9x-11z+10=0\end{array}\right\}$

12. $\left.\begin{array}{l}x+2y+3z=6\\2x+4y+2z=8\\3x+2y+8z=101\end{array}\right\}$

13. $\left.\begin{array}{l}x-3y-2z=1\\2x-3y+5z=-19\\5x+2y-z=12\end{array}\right\}$

14. $\left.\begin{array}{l}3x-2y=5\\4x-3y+2z=11\\x-2y-5z=-7\end{array}\right\}$

15. $\left.\begin{array}{l}x+y=1\\y+z=9\\x+z=5\end{array}\right\}$

16. $\left.\begin{array}{l}2x-3y=3\\3y-4z=7\\4z-5x=2\end{array}\right\}$

17. $\left.\begin{array}{l}3x-4y+6z=1\\2x+2y-z=1\\7x-6y+7z=2\end{array}\right\}$

18. $\left.\begin{array}{l}7x-3y=30\\9y-5z=34\\x+y+z=33\end{array}\right\}$

19. $$\left.\begin{aligned} x+\frac{y}{2}+\frac{z}{3}&=6\\ y+\frac{z}{2}+\frac{x}{3}&=-1\\ z+\frac{x}{2}+\frac{y}{3}&=17 \end{aligned}\right\}$$

20. $$\left.\begin{aligned} \frac{1}{x}+\frac{2}{y}&=5\\ \frac{3}{y}-\frac{4}{z}&=-6\\ \frac{3}{z}-\frac{4}{x}&=5 \end{aligned}\right\}$$

21. $$\left.\begin{aligned} \frac{1}{x}+\frac{1}{y}-\frac{1}{z}&=a\\ \frac{1}{x}-\frac{1}{y}+\frac{1}{z}&=b\\ \frac{1}{y}+\frac{1}{z}-\frac{1}{x}&=c \end{aligned}\right\}*$$

22. $$\left.\begin{aligned} bz+cy&=a\\ az+cx&=b\\ ay+bx&=c \end{aligned}\right\}\dagger$$

23. $$\left.\begin{aligned} \frac{3}{x}-\frac{4}{5y}+\frac{1}{z}&=7\tfrac{3}{5}\\ \frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}&=10\tfrac{1}{6}\\ \frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}&=16\tfrac{1}{10} \end{aligned}\right\}$$

24. $$\left.\begin{aligned} \frac{2}{x}-\frac{3}{y}+\frac{4}{z}&=2.9\\ \frac{5}{x}-\frac{6}{y}-\frac{7}{z}&=-10.4\\ \frac{9}{y}+\frac{10}{z}-\frac{8}{x}&=14.9 \end{aligned}\right\}$$

25. $$\left.\begin{aligned} \frac{2}{x}+\frac{1}{y}-\frac{3}{z}&=0\\ \frac{3}{z}-\frac{2}{y}-2&=0\\ \frac{1}{x}+\frac{1}{z}-\frac{4}{3}&=0 \end{aligned}\right\}$$

26. $$\left.\begin{aligned} ax+by+cz&=a\\ ax-by-cz&=b\\ ax+cy+bz&=c \end{aligned}\right\}$$

27. $$\frac{2x-y}{3}=\frac{3y+2z}{4}=\frac{x-y-z}{5}=4$$

28. $$\frac{x-y}{a}=\frac{y-z}{b}=\frac{x+z}{c}=\frac{x-a-b}{a+b+c}$$

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* Subtract from the sum of the three equations each equation separately.

† Multiply the equations by a, b, and c, respectively, and from the sum of the results subtract the double of each equation separately.

CHAPTER XII.

Problems producing Simultaneous Equations.

197. It is often necessary in the solution of problems to employ two or more letters to represent the quantities to be found. In all cases the conditions must be sufficient to give just as many equations as there are unknown quantities employed.

If there be *more* equations than unknown quantities, some of them are superfluous or contradictory; if there be *less* equations than unknown quantities, the problem is indeterminate or impossible.

(1) When the greater of two numbers is divided by the less the quotient is 4 and the remainder 3; and when the sum of the two numbers is increased by 38, and the result divided by the greater of the two numbers, the quotient is 2 and the remainder 2. Find the numbers.

Let $x =$ the greater number,

and $y =$ the smaller number.

Then $$\frac{x-3}{y} = 4,$$

and $$\frac{x+y+38-2}{x} = 2.$$

From the solution of these equations $x = 47$, and $y = 11$.

(2) If A give B $10, B will have three times as much money as A. If B give A $10, A will have twice as much money as B. How much has each?

Let $\qquad x =$ number of dollars A has,

and $\qquad y =$ number of dollars B has.

Then $y + 10 =$ number of dollars B has, and $x - 10 =$ number of dollars A has after A gives $10 to B.

$\therefore y + 10 = 3(x - 10)$, and $x + 10 = 2(y - 10)$.

From the solution of these equations, $x = 22$ and $y = 26$.

Therefore, A has $22 and B $26.

Exercise LXXV.

1. The sum of two numbers divided by 2 gives as a quotient 24, and the difference between them divided by 2 gives as a quotient 17. What are the numbers?

2. The number 144 is divided into three numbers. When the first is divided by the second, the quotient is 3 and the remainder 2; and when the third is divided by the sum of the other two numbers, the quotient is 2 and the remainder 6. Find the numbers.

3. Three times the greater of two numbers exceeds twice the less by 10; and twice the greater together with three times the less is 24. Find the numbers.

4. If the smaller of two numbers be divided by the greater, the quotient is .21 and the remainder .0057; but if the greater be divided by the smaller, the quotient is 4 and the remainder .742. What are the numbers?

5. Seven years ago the age of a father was four times that of his son; seven years hence the age of the father will be double that of the son. What are their ages?

6. The sum of the ages of a father and son is half what it will be in 25 years; the difference between their ages is one-third of what the sum will be in 20 years. What are their ages?

7. If B give A $25, they will have equal sums of money; but if A give B $22, B's money will be double that of A. How much has each?

8. A farmer sold to one person 30 bushels of wheat and 40 bushels of barley for $67.50; to another person he sold 50 bushels of wheat and 30 bushels of barley for $85. What was the price of the wheat and of the barley per bushel?

9. If A give B $5, he will then have $6 less than B; but if he receive $5 from B, three times his money will be $20 more than four times B's. How much has each?

10. The cost of 12 horses and 14 cows is $1900; the cost of 5 horses and 3 cows is $650. What is the cost of a horse and a cow respectively?

NOTE I. A fraction of which the terms are unknown may be represented by $\frac{x}{y}$.

Ex. A certain fraction becomes equal to $\frac{1}{2}$ if 3 be added to its numerator, and equal to $\frac{2}{7}$ if 3 be added to its denominator. Determine the fraction.

Let $\frac{x}{y}$ = the required fraction.

By the conditions $\frac{x+3}{y} = \frac{1}{2}$,

and $\frac{x}{y+3} = \frac{2}{7}$.

From the solution of these equations it is found that

$$x = 6,$$
$$y = 18.$$

Therefore the fraction $= \frac{6}{18}$.

11. A certain fraction becomes equal to 2 when 7 is added to its numerator, and equal to 1 when 1 is subtracted from its denominator. Determine the fraction.

12. A certain fraction becomes equal to $\frac{1}{2}$ when 7 is added to its denominator, and equal to 2 when 13 is added to its numerator. Determine the fraction.

13. A certain fraction becomes equal to $\frac{7}{9}$ when the denominator is increased by 4, and equal to $\frac{29}{41}$ when the numerator is diminished by 15. Determine the fraction.

14. A certain fraction becomes equal to $\frac{4}{5}$ if 7 be added to the numerator, and equal to $\frac{3}{8}$ if 7 be subtracted from the denominator. Determine the fraction.

15. Find two fractions with numerators 2 and 5 respectively, whose sum is $1\frac{1}{2}$, and if their denominators are interchanged their sum is 2.

16. A fraction which is equal to $\frac{2}{3}$ is increased to $\frac{8}{11}$ when a certain number is added to both its numerator and denominator, and is diminished to $\frac{5}{8}$ when one more than the same number is subtracted from each. Determine the fraction.

Note II. A number consisting of *two* digits which are unknown may be represented by $10x + y$, in which x and y represent the digits of the number. Likewise, a number consisting of *three* digits which are unknown may be represented by $100x + 10y + z$, in which x, y, and z represent the digits of the number.

For example, consider any number expressed by three digits, as 364. The expression 364 means $300 + 60 + 4$; or, 100 *times* $3 + 10$ *times* $6 + 4$.

Ex. The sum of the two digits of a number is 8, and if 36 be added to the number the digits will be interchanged. What is the number?

Let $x =$ the digit in the tens' place,

and $y =$ the digit in the units' place.

Then $10x + y =$ the number.

By the conditions, $x + y = 8,$ (1)

and $10x + y + 36 = 10y + x.$ (2)

From (2), $9x - 9y = -36.$
Divide by 9, $x - y = -4.$
Add (1) and (3), $2x = 4,$
$\therefore x = 2.$
Subtract (3) from (1), $2y = 12,$
$\therefore y = 6.$

Hence, the number is 26.

17. The sum of the two digits of a number is 10, and if 54 be added to the number the digits will be interchanged. What is the number?

18. The sum of the two digits of a number is 6, and if the number be divided by the sum of the digits the quotient is 4. What is the number?

19. A certain number is expressed by two digits, of which the first is the greater. If the number be divided by the sum of its digits the quotient is 7; if the digits be interchanged, and the resulting number diminished by 12 be divided by the difference between the two digits, the quotient is 9. What is the number?

20. If a certain number be divided by the sum of its two digits the quotient is 6 and the remainder 3; if the digits be interchanged, and the resulting number be divided by the sum of the digits, the quotient is 4 and the remainder 9. What is the number?

21. If a certain number be divided by the sum of its two digits diminished by 2, the quotient is 5 and the remainder 1; if the digits be interchanged, and the resulting number be divided by the sum of the digits increased by 2, the quotient is 5 and the remainder 8. Find the number.

22. The first of the two digits of a number is, when doubled, 3 more than the second, and the number itself is less by 6 than five times the sum of the digits. What is the number?

23. A number is expressed by three digits, of which the first and last are alike. By interchanging the digits in the units' and tens' places the number is increased by 54; but if the digits in the tens' and hundreds' places are interchanged, 9 must be added to four times the resulting number to make it equal to the original number. What is the number?

24. A number is expressed by three digits. The sum of the digits is 21; the sum of the first and second exceeds the third by 3; and if 198 be added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

25. A number is expressed by three digits. The sum of the digits is 9; the number is equal to forty-two times the sum of the first and second digits; and the third digit is twice the sum of the other two. Find the number.

26. A certain number, expressed by three digits, is equal to forty-eight times the sum of its digits. If 198 be subtracted from the number, the digits in the units' and hundreds' places will be interchanged; and the sum of the extreme digits is equal to twice the middle digit. Find the number.

NOTE III. If a boat move at the rate of x miles an hour in still water, and if it be on a stream that runs at the rate of y miles an hour, then

$x + y$ represents its rate *down* the stream,
$x - y$ represents its rate *up* the stream.

27. A waterman rows 30 miles and back in 12 hours. He finds that he can row 5 miles with the stream in the same time as 3 against it. Find the time he was rowing up and down respectively.

28. A crew which can pull at the rate of 12 miles an hour down the stream, finds that it takes twice as long to come up the river as to go down. At what rate does the stream flow?

29. A man sculls down a stream, which runs at the rate of 4 miles an hour, for a certain distance in 1 hour and 40 minutes. In returning it takes him 4 hours and 15 minutes to arrive at a point 3 miles short of his starting-place. Find the distance he pulled down the stream and the rate of his pulling.

30. A person rows down a stream a distance of 20 miles and back again in 10 hours. He finds he can row 2 miles against the stream in the same time he can row 3 miles with it. Find the time of his rowing down and of his rowing up the stream; and also the rate of the stream.

NOTE IV. When commodities are mixed, it is to be observed that the quantity of the mixture = the quantity of the ingredients; the cost of the mixture = the cost of the ingredients.

Ex. A wine-merchant has two kinds of wine which cost 72 cents and 40 cents a quart respectively. How much of each must he take to make a mixture of 50 quarts worth 60 cents a quart?

Let x = required number of quarts worth 72 cents a quart,

and y = required number of quarts worth 40 cents a quart.

Then, $72x$ = cost in cents of the first kind,

$40y$ = cost in cents of the second kind of wine,

and 3000 = cost in cents of the mixture.

$$\therefore x + y = 50,$$
$$72x + 40y = 3000.$$

From which equations the values of x and y may be found.

31. A grocer mixed tea that cost him 42 cents a pound with tea that cost him 54 cents a pound. He had 30 pounds of the mixture, and by selling it at the rate of 60 cents a pound, he gained as much as 10 pounds of the cheaper tea cost him. How many pounds of each did he put into the mixture?

32. A grocer mixes tea that cost him 90 cents a pound with tea that cost him 28 cents a pound. The cost of the mixture is $61.20. He sells the mixture at 50 cents a pound, and gains $3.80. How many pounds of each did he put into the mixture?

33. A farmer has 28 bushels of barley worth 84 cents a bushel. With his barley he wishes to mix rye worth $1.08 a bushel, and wheat worth $1.44 a bushel, so that the mixture may be 100 bushels, and be worth $1.20 a bushel. How many bushels of rye and of wheat must he take?

NOTE V. It is to be remembered that if a person can do a piece of work in x days, *the part* of the work he can do in *one* day will be represented by $\frac{1}{x}$.

Ex. A and B together can do a piece of work in 48 days; A and C together can do it in 30 days; B and C together can do it in $26\frac{2}{3}$ days. How long will it take each to do the work?

Let $x =$ the number of days it will take A alone to do the work,
$y =$ the number of days it will take B alone to do the work,
and $z =$ the number of days it will take C alone to do the work.

Then, $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, respectively, will denote the part each can do in a day,

and $\frac{1}{x} + \frac{1}{y}$ will denote the part A and B together can do in a day;

but $\frac{1}{48}$ will denote the part A and B together can do in a day.

$$\text{Therefore,} \qquad \frac{1}{x}+\frac{1}{y}=\frac{1}{48} \qquad (1)$$

$$\text{Likewise,} \qquad \frac{1}{x}+\frac{1}{z}=\frac{1}{30} \qquad (2)$$

$$\text{and} \qquad \frac{1}{y}+\frac{1}{z}=\frac{1}{26\frac{2}{3}}=\frac{3}{80} \qquad (3)$$

$$\text{Add (1), (2), and (3),} \qquad \frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{11}{120} \qquad (4)$$

$$\text{Multiply (1) by 2,} \qquad \frac{2}{x}+\frac{2}{y}=\frac{1}{24} \qquad (5)$$

$$\text{Subtract (5) from (4),} \qquad \frac{2}{z}=\frac{1}{20}$$

$$\therefore\ z=40.$$

$$\text{Subtract the double of (2) from (4),} \qquad \frac{2}{y}=\frac{1}{40}$$

$$\therefore\ y=80.$$

$$\text{Subtract the double of (3) from (4),} \qquad \frac{2}{x}=\frac{1}{60}$$

$$\therefore\ x=120.$$

34. A and B together earn \$40 in 6 days; A and C together earn \$54 in 9 days; B and C together earn \$80 in 15 days. What does each earn a day?

35. A cistern has three pipes, A, B, and C. A and B will fill it in 1 hour and 10 minutes; A and C in 1 hour and 24 minutes; B and C in 2 hours and 20 minutes. How long will it take each to fill it?

36. A warehouse will hold 24 boxes and 20 bales; 6 boxes and 14 bales will fill half of it. How many of each alone will it hold?

37. Two workmen together complete some work in 20 days; but if the first had worked twice as fast, and the second half as fast, they would have finished it in 15 days. How long would it take each alone to do the work?

38. A purse holds 19 crowns and 6 guineas; 4 crowns and 5 guineas fill $\frac{17}{68}$ of it. How many of each alone will it hold?

39. A piece of work can be completed by A, B, and C together in 10 days; by A and B together in 12 days; by B and C, if B work 15 days and C 30 days. How long will it take each alone to do the work?

40. A cistern has three pipes, A, B, and C. A and B will fill it in a minutes; A and C in b minutes; B and C in c minutes. How long will it take each alone to fill it?

NOTE VI. In considering *the rate of increase or decrease* in quantities, it is usual to take 100 as a *common standard of reference*, so that the increase or decrease is calculated for every 100, and therefore called *per cent.*

It is to be observed that the representative of the number resulting after an increase has taken place is 100 + increase per cent; and after a decrease, 100 − decrease per cent.

Interest depends upon the *time* for which the money is lent, as well as upon the *rate per cent* charged; the rate per cent charged being the rate per cent on the principal for *one year.* Hence,

$$\text{Simple interest} = \frac{\text{Principal} \times \text{Rate} \times \text{Time}}{100},$$

where Time means *number of years or fraction of a year.*

$$\text{Amount} = \text{Principal} + \text{Interest}.$$

In questions relating to stocks, 100 is taken as the representative of the *stock*, the *price* represents its market value, and the *per cent* represents the *interest* which the *stock* bears. Thus, if six per cent stocks are quoted at 108, the meaning is, that the price of \$100 of the stock is \$108, and that the interest derived from \$100 of the *stock* will be $\frac{6}{100}$ of \$100, that is, \$6 a year. The rate of interest on the *money invested* will be $\frac{100}{108}$ of 6 per cent.

41. A man has \$10,000 invested. For a part of this sum he receives 5 per cent interest, and for the rest 4 per cent; the income from his 5 per cent investment is \$50 more than from his 4 per cent. How much has he in each investment?

42. A sum of money, at simple interest, amounted in 6 years to \$26,000, and in 10 years to \$30,000. Find the sum and the rate of interest.

43. A sum of money, at simple interest, amounted in 10 months to \$26,250, and in 18 months to \$27,250. Find the sum and the rate of interest.

44. A sum of money, at simple interest, amounted in m years to a dollars, and in n years to b dollars. Find the sum and the rate of interest.

45. A sum of money, at simple interest, amounted in a months to c dollars, and in b months to d dollars. Find the sum and the rate of interest.

46. A person has a certain capital invested at a certain rate per cent. Another person has \$1000 more capital, and his capital invested at one per cent better than the first, and receives an income \$80 greater. A third person has \$1500 more capital, and his capital invested at two per cent better than the first, and receives an income \$150 greater. Find the capital of each, and the rate at which it is invested.

47. A person has \$12,750 to invest. He can buy three per cent bonds at 81, and five per cents at 120. Find the amount of money he must invest in each in order to have the same income from each investment.

48. A and B each invested \$1500 in bonds; A in three per cents and B in four per cents. The bonds were bought at such prices that B received \$5 interest more than A. After both classes of bonds rose 10 points, they sold out, and A received \$50 more than B. What price was paid for each class of bonds?

49. A person invests \$10,000 in three per cent bonds, \$16,500 in three and one-half per cents, and has an income from both investments of \$1056.25. If his investments had been \$2750 more in the three per cents, and less in the three and one-half per cents, his income would have been $62\frac{1}{2}$ cents greater. What price was paid for each class of bonds?

50. The sum of \$2500 was divided into two unequal parts and invested, the smaller part at two per cent more than the larger. The *rate* of interest on the larger sum was afterwards increased by 1, and that of the smaller sum diminished by 1; and thus the *interest* of the whole was increased by one-fourth of its value. If the interest of the larger sum had been so increased, and no change been made in the interest of the smaller sum, the interest of the whole would have been increased one-third of its value. Find the sums invested, and the rate per cent of each.

NOTE VII. If x represent the number of linear units in the length, and y in the width, of a rectangle, xy will represent the number of its units of surface; the surface unit having the same name as the linear unit of its sides.

51. If the sides of a rectangular field were each increased by 2 yards, the area would be increased by 220 square yards; if the length were increased and the breadth were diminished each by 5 yards, the area would be diminished by 185 square yards. What is its area?

52. If a given rectangular floor had been 3 feet longer and 2 feet broader it would have contained 64 square feet more; but if it had been 2 feet longer and 3 feet broader it would have contained 68 square feet more. Find the length and breadth of the floor.

53. In a certain rectangular garden there is a strawberry-bed whose sides are one-third of the lengths of the corresponding sides of the garden. The perimeter of the garden exceeds that of the bed by 200 yards; and if the greater side of the garden be increased by 3, and the other by 5 yards, the garden will be enlarged by 645 square yards. Find the length and breadth of the garden.

NOTE VIII. Care must be taken to express the conditions of a problem with reference to the same principal unit.

Ex. In a mile race A gives B a start of 20 yards and beats him by 30 seconds. At the second trial A gives B a start of 32 seconds and beats him by $9\frac{5}{11}$ yards. Find the rate per hour at which each runs.

Let $x =$ number of yards A runs a second,

and $y =$ number of yards B runs a second.

Since there are 1760 yards in a mile,

$\frac{1760}{x} =$ number of seconds it takes A to run a mile,

$\frac{1740}{y}$ and $\frac{1750\frac{6}{11}}{y} =$ number of seconds B was running in the first and second trials, respectively.

$$\text{Hence, } \frac{1740}{y} - \frac{1760}{x} = 30,$$

$$\text{and } \frac{1750\frac{6}{11}}{y} - \frac{1760}{x} = 32.$$

The solution of these equations gives $x = 5\frac{13}{15}$ and $y = 5\frac{3}{11}$.

That is, A runs $\frac{5\frac{13}{15}}{1760}$, or $\frac{1}{300}$, of a mile in one second;

and in one hour, or 3600 seconds, runs 12 miles.

Likewise, B runs $10\frac{95}{121}$ miles in one hour.

54. In a mile race A gives B a start of 100 yards and beats him by 15 seconds. In the second trial A gives B a start of 45 seconds and is beaten by 22 yards. Find the rate of each in miles per hour.

55. In a mile race A gives B a start of 44 yards and beats him by 51 seconds. In the second trial A gives B a start of 1 minute and 15 seconds and is beaten by 88 yards. Find the rate of each in miles per hour.

56. The time which an express-train takes to go 120 miles is $\frac{9}{14}$ of the time taken by an accommodation-train. The slower train loses as much time in stopping at different stations as it would take to travel 20 miles without stopping; the express-train loses only half as much time by stopping as the accommodation-train, and travels 15 miles an hour faster. Find the rate of each train in miles per hour.

57. A train moves from P towards Q, and an hour later a second train starts from Q and moves towards P at a rate of 10 miles an hour more than the first train; the trains meet half-way between P and Q. If the train from P had started an hour after the train from Q its rate must have been increased by 28 miles in order that the trains should meet at the half-way point. Find the distance from P to Q.

58. A passenger-train, after travelling an hour, meets with an accident which detains it one-half an hour; after which it proceeds at four-fifths of its usual rate, and arrives an hour and a quarter late. If the accident had happened 30 miles farther on, the train would have been only an hour late. Determine the usual rate of the train.

59. A passenger-train after travelling an hour is detained 15 minutes; after which it proceeds at three-fourths of its former rate, and arrives 24 minutes late. If the detention had taken place 5 miles farther on, the train would have been only 21 minutes late. Determine the usual rate of the train.

60. A man bought 10 oxen, 120 sheep, and 46 lambs. The cost of 3 sheep was equal to that of 5 lambs; an ox, a sheep, and a lamb together cost a number of dollars less by 57 than the whole number of animals bought; and the whole sum spent was \$2341.50. Find the price of an ox, a sheep, and a lamb, respectively.

61. A farmer sold 100 head of stock, consisting of horses, oxen, and sheep, so that the whole realized \$11.75 a head; while a horse, an ox, and a sheep were sold for \$110, \$62.50, and \$7.50, respectively. Had he sold one-fourth of the number of oxen that he did, and 25 more sheep, he would have received the same sum. Find the number of horses, oxen, and sheep, respectively, which were sold.

62. A, B, and C together subscribed \$100. If A's subscription had been one-tenth less, and B's one-tenth more, C's must have been increased by \$2 to make up the sum; but if A's had been one-eighth more, and B's one-eighth less, C's subscription would have been \$17.50. What did each subscribe?

63. A gives to B and C as much as each of them has; B gives to A and C as much as each of them then has; and C gives to A and B as much as each of them then has. In the end each of them has \$6. How much had each at first?

64. A pays to B and C as much as each of them has; B pays to A and C one-half as much as each of them then has; and C pays to A and B one-third of what each of them then has. In the end A finds that he has \$1.50, B \$4.16$\frac{2}{3}$, C \$.58$\frac{1}{3}$. How much had each at first?

CHAPTER XIII.

INVOLUTION AND EVOLUTION.

198. The operation of raising an expression to any required *power* is called **Involution.**

Every case of involution is merely an example of *multiplication,* in which the factors are *equal.* Thus,

$$(2a^3)^2 = 2a^3 \times 2a^3 = 4a^6.$$

199. A power of a simple expression is found by multiplying the exponent of each factor by the exponent of the required factor, and taking the product of the resulting factors. The proof of the law of exponents, in its general form, is:

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times a^m \times \ldots\ldots \text{ to } n \text{ factors,}\\ &= a^{m+m+m+\ldots\ldots \text{ to } n \text{ terms}},\\ &= a^{mn}.\end{aligned}$$

Hence, if the exponent of the required power be a composite number, it may be resolved into prime factors, the power denoted by one of these factors may be found, and the result raised to a power denoted by another, and so on. Thus, the fourth power may be obtained by taking the second power of the second power; the sixth by taking the second power of the third power; the eighth by taking the second power of the second power of the second power.

200. From the **Law of Signs** in multiplication it is evident that,

I. All *even* powers of a number are *positive.*

II. All *odd* powers of a number have the *same sign* as the number itself.

Hence, no *even* power of *any* number can be *negative*; and of two compound expressions whose terms are identical but have opposite signs, the even powers are the same. Thus,

$$(b-a)^2 = \{-(a-b)\}^2 = (a-b)^2.$$

201. A method has been given, § 83, of finding, without actual multiplication, the powers of binomials which have the form $(a \pm b)$.

The same method may be employed when the terms of a binomial have *coefficients* or *exponents*.

(1) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$

(2) $(5x^2 - 2y^3)^3,$

$$= (5x^2)^3 - 3(5x^2)^2(2y^3) + 3(5x^2)(2y^3)^2 - (2y^3)^3,$$
$$= 125x^6 - 150x^4y^3 + 60x^2y^6 - 8y^9.$$

(3) $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$

(4) $(x^2 - \frac{1}{2}y)^4,$

$$= (x^2)^4 - 4(x^2)^3(\tfrac{1}{2}y) + 6(x^2)^2(\tfrac{1}{2}y)^2 - 4x^2(\tfrac{1}{2}y)^3 + (\tfrac{1}{2}y)^4,$$
$$= x^8 - 2x^6y + \tfrac{3}{2}x^4y^2 - \tfrac{1}{2}x^2y^3 + \tfrac{1}{16}y^4.$$

202. In like manner, a *polynomial* of three or more terms may be raised to any power by enclosing its terms in parentheses, so as to give the expression the form of a binomial. Thus,

(1) $(a+b+c)^3 = \{a + (b+c)\}^3,$

$$= a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3,$$
$$= a^3 + 3a^2b + 3a^2c + 3ab^2 + 6abc$$
$$+ 3ac^2 + b^3 + 3b^2c + 3bc^2 + c^3.$$

(2) $(x^3 - 2x^2 + 3x + 4)^2$,

$$= \{(x^3 - 2x^2) + (3x + 4)\}^2,$$
$$= (x^3 - 2x^2)^2 + 2(x^3 - 2x^2)(3x + 4) + (3x + 4)^2,$$
$$= x^6 - 4x^5 + 4x^4 + 6x^4 - 4x^3 - 16x^2 + 9x^2 + 24x + 16,$$
$$= x^6 - 4x^5 + 10x^4 - 4x^3 - 7x^2 + 24x + 16.$$

Exercise LXXVI.

Write the second members of the following equations:

1. $(a^3)^2 =$	11. $(2a^2bc^3)^4 =$	21. $(-3a^2b^2c)^5 =$
2. $(x^5)^3 =$	12. $(-5ax^3y^2)^3 =$	22. $(-3xy^3)^6 =$
3. $(x^2y^3)^2 =$	13. $(-7m^3nx^2y^4)^2 =$	23. $(-5a^2bx^3)^5 =$
4. $\left(\frac{a^3b^2}{2}\right)^4 =$	14. $\left(-\frac{2x^3y}{3abc}\right)^5 =$	24. $\left(-\frac{3ab^2}{4c^3}\right)^4 =$
5. $\left(\frac{3x^2y}{2a^3b^2}\right)^5 =$	15. $(3x + 1)^4 =$	25. $\left(-\frac{x^3y^3z^4}{2}\right)^7 =$
6. $(x + 2)^3 =$	16. $(2x - a)^4 =$	26. $(1 - a - a^2)^2 =$
7. $(x - 2)^4 =$	17. $(3x + 2a)^5 =$	27. $(2 - 3x + 4x^2)^3 =$
8. $(x + 3)^5 =$	18. $(2x - y)^4 =$	28. $(1 - 2x + x^2)^3 =$
9. $(1 + 2x)^5 =$	19. $(x^2y - 2xy^2)^6 =$	29. $(1 - x + x^2)^3 =$
10. $(2m - 1)^3 =$	20. $(ab - 3)^7 =$	30. $(1 + x + x^2)^4 =$

Evolution.

203. The operation of finding any required *root* of an expression is called **Evolution.**

Every case of evolution is merely an example of *factoring*, in which the required factors are all *equal*. Thus, the square, cube, fourth..... roots of an expression are found by taking one of the *two*, *three*, *four*..... *equal factors* of the expression.

204. The symbol which denotes that a square root is to be extracted is $\sqrt{\ }$; and for other roots the same symbol is used, but with a figure written above to indicate the root, thus, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, etc., signifies the *third* root, *fourth* root, etc.

205. Since the *cube* of $a^2 = a^6$, the *cube root* of $a^6 = a^2$.

Since the *fourth power* of $2a^2 = 2^4a^8$, the *fourth root* of $2^4a^8 = 2a^2$.

Since the square of $abc = a^2b^2c^2$, the *square root* of $a^2b^2c^2 = abc$.

Since the *square* of $\dfrac{ab}{xy} = \dfrac{a^2b^2}{x^2y^2}$, the *square root* of $\dfrac{a^2b^2}{x^2y^2} = \dfrac{ab}{xy}$.

Hence, the root of a simple expression is found by *dividing the exponent of each factor by the index of the root, and taking the product of the resulting factors.*

206. It is evident from § 200 that

I. Any *even* root of a *positive* number will have the double sign, $\pm$.

II. There can be no *even* root of a *negative* number.

III. Any *odd* root of a number will have the same sign as the number.

Thus, $\sqrt{\dfrac{16x^2}{81y^2}} = \pm\dfrac{4x}{9y}$; $\sqrt[3]{-27m^3n^6} = -3mn^2$;

$$\sqrt[4]{\frac{16x^8y^{12}}{81a^{16}}} = \pm\frac{2x^2y^3}{3a^4}.$$

But $\sqrt{-x^2}$ is neither $+x$ nor $-x$, for $(+x)^2 = +x^2$, and $(-x)^2 = +x^2$.

The indicated even root of a negative number is called an **impossible**, or **imaginary**, number.

207. If the root of a number expressed in figures is not readily detected, it may be found by resolving the number into its prime factors. Thus, to find the square root of 3,415,104:

$$\begin{array}{r|r}
2^3 & 3415104 \\ \hline
2^3 & 426888 \\ \hline
3^2 & 53361 \\ \hline
7 & 5929 \\ \hline
7 & 847 \\ \hline
11 & 121 \\ \hline
 & 11
\end{array}$$

$$\therefore \quad 3{,}415{,}104 = 2^6 \times 3^2 \times 7^2 \times 11^2.$$
$$\therefore \sqrt{3{,}415{,}104} = 2^3 \times 3 \times 7 \times 11 = 1848.$$

Exercise LXXVII.

Simplify:

1. $\sqrt{a^4}, \sqrt[4]{x^8}, \sqrt{4a^6b^2}, \sqrt[3]{64}, \sqrt[5]{a^5x^{10}y^{15}}, \sqrt[4]{16a^{12}b^4c^8}, \sqrt[5]{-32a^{15}}.$

2. $\sqrt[3]{-1728c^6d^{12}x^3y^9},\ \sqrt[3]{3375b^{21}z^{15}},\ \sqrt[4]{3111696c^{16}z^4}.$

3. $\sqrt{53361b^4c^8y^{12}z^{16}},\ \sqrt[3]{-\dfrac{216b^3c^{15}}{343z^{24}}},\ \sqrt[6]{\dfrac{64x^{18}}{729z^{30}}}.$

4. $\sqrt{25a^2b^4c^2} + \sqrt[3]{8a^3b^6c^3} - \sqrt[4]{81a^4b^8c^4} - \sqrt[5]{32a^5b^{10}c^5}.$

5. $\sqrt[3]{27x^3y^6} \times \sqrt[5]{243y^5z^5} \times \sqrt{16x^4z^2}.$

When $a = 1$, $b = 3$, $x = 2$, $y = 6$, find the values of:

6. $4\sqrt{2x} - \sqrt{abxy} + 5\sqrt{a^2b^3xy}.$

7. $2a\sqrt{8ax} + b\sqrt[3]{12by} + 4abx\sqrt{bxy}.$

8. $\sqrt{a^2 + 2ab + b^2} \times \sqrt[3]{a^3 + 3a^2b + 3ab^2 + b^3}.$

9. $\sqrt[3]{b^3 - 3b^2a + 3ba^2 - a^3} \div \sqrt{b^2 + a^2 - 2ab}.$

SQUARE ROOTS OF COMPOUND EXPRESSIONS.

208. Since the square of $a+b$ is $a^2+2ab+b^2$, the square root of $a^2+2ab+b^2$ is $a+b$.

It is required to find a method of extracting the root $a+b$ when $a^2+2ab+b^2$ is given:

$$\begin{array}{r|l}
 & a^2+2ab+b^2 \,\underline{|\, a+b} \\
 & a^2 \\
2a+b & 2ab+b^2 \\
 & \underline{2ab+b^2}
\end{array}$$

Ex. The first term, a, of the root is obviously the square root of the first term, a^2, in the expression.

If the a^2 be subtracted from the given expression, the remainder is $2ab+b^2$. Therefore the second term, b, of the root is obtained when the first term of this remainder is divided by $2a$, that is, by *double the part of the root already found.* Also, since $2ab+b^2=(2a+b)b$, the divisor *is completed by adding to the trial-divisor the new term of the root.*

(1) Find the square root of $25x^2-20x^3y+4x^4y^2$.

$$\begin{array}{r|l}
 & 25x^2-20x^3y+4x^4y^2 \,\underline{|\, 5x-2x^2y} \\
 & 25x^2 \\
10x-2x^2y & -20x^3y+4x^4y^2 \\
 & \underline{-20x^3y+4x^4y^2}
\end{array}$$

The expression is *arranged* according to the ascending powers of x.

The square root of the first term is $5x$, and $5x$ is placed at the right of the given expression, for the first term of the root.

The second term of the root, $-2x^2y$, is obtained by dividing $-20x^3y$ by $10x$, and this new term of the root is also annexed to the divisor, $10x$, to complete the divisor.

209. The same method will apply to longer expressions, if care be taken to obtain the *trial-divisor* at each stage of the process, by *doubling the part of the root already found,* and to obtain the *complete divisor by annexing the new term of the root to the trial-divisor.*

Ex. Find the square root of

$$1+10x^2+25x^4+16x^6-24x^5-20x^3-4x.$$

$$\begin{array}{l|l|l}
 & 16x^6-24x^5+25x^4-20x^3+10x^2-4x+1 & 4x^3-3x^2+2x-1 \\
 & 16x^6 & \\
8x^3-3x^2 & -24x^5+25x^4 & \\
 & -24x^5+9x^4 & \\
8x^3-6x^2+2x & 16x^4-20x^3+10x^2 & \\
 & 16x^4-12x^3+4x^2 & \\
8x^3-6x^2+4x-1 & -8x^3+6x^2-4x+1 & \\
 & -8x^3+6x^2-4x+1 &
\end{array}$$

The expression is arranged according to the descending powers of x.

It will be noticed that each successive trial-divisor may be obtained by taking the preceding complete divisor with its *last term doubled.*

Exercise LXXVIII.

Extract the square roots of:

1. $a^4+4a^3+2a^2-4a+1.$
2. $x^4-2x^3y+3x^2y^2-2xy^3+y^4.$
3. $4a^6-12a^5x+5a^4x^2+6a^3x^3+a^2x^4.$
4. $9x^6-12x^3y^3+16x^2y^4-24x^4y^2+4y^6+16xy^5.$
5. $4a^8+16c^8+16a^6c^2-32a^2c^6.$
6. $4x^4+9-30x-20x^3+37x^2.$
7. $16x^4-16abx^2+16b^2x^2+4a^2b^2-8ab^3+4b^4.$
8. $x^6+25x^2+10x^4-4x^5-20x^3+16-24x.$
9. $x^6+8x^4y^2-4x^5y-4xy^5+8x^2y^4-10x^3y^3+y^6.$
10. $4-12a-11a^4+5a^2-4a^5+4a^6+14a^3.$
11. $9a^2-6ab+30ac+6ad+b^2-10bc-2bd$
 $+25c^2+10cd+d^2.$

12. $25x^6 - 31x^4y^2 + 34x^3y^3 - 30x^5y + y^6 - 8xy^5 + 10x^2y^4.$

13. $m^8 - 4m^7 + 10m^6 - 20m^5 - 44m^3$
$+ 35m^4 + 46m^2 - 40m + 25.$

14. $x^4 - x^3y - \frac{7}{4}x^2y^2 + xy^3 + y^4.$

15. $x^4 - 4x^3y + 6x^2y^2 - 6xy^3 + 5y^4 - \frac{2y^5}{x} + \frac{y^6}{x^2}.$

16. $\frac{a^4}{9} - \frac{a^3x}{2} + \frac{43}{48}a^2x^2 - \frac{3}{4}ax^3 + \frac{x^4}{4}.$

17. $1 + \frac{4}{x} + \frac{10}{x^2} + \frac{20}{x^3} + \frac{25}{x^4} + \frac{24}{x^5} + \frac{16}{x^6}.$

18. $\frac{a^2}{b^2} - \frac{2a}{b} + 3 - \frac{2b}{a} + \frac{b^2}{a^2}.$

19. $x^4 + x^3 - \frac{5x^2}{12} - \frac{x}{3} + \frac{1}{9}.$

Square Roots of Arithmetical Numbers.

210. In the general method of extracting the square root of a number expressed by figures, the first step is to mark off the figures in *periods*.

Since $1 = 1^2$, $100 = 10^2$, $10,000 = 100^2$, and so on, it is evident that the square root of any number between 1 and 100 lies between 1 and 10; the square root of any number between 100 and 10,000 lies between 10 and 100. In other words, the square root of any number expressed by *one* or *two* figures is a number of *one* figure; the square root of any number expressed by *three* or *four* figures is a number of *two* figures; and so on.

If, therefore, a dot be placed over the *units' figure* of a square number, and over every *alternate* figure, the number of dots will be equal to the number of figures in its square root.

Find the square root of 3249.

$$\begin{array}{r|l} & 3\dot{2}4\dot{9}\,(57 \\ & 25 \\ 107 & 749 \\ & 749 \end{array}$$

In this case, a in the typical form $a^2 + 2ab + b^2$ represents 5 *tens*, that is, 50, and b represents 7. The 25 subtracted is really 2500, that is, a^2, and the complete divisor, $2a + b$, is $2 \times 50 + 7 = 107$.

211. The same method will apply to numbers of more than two periods by considering a in the typical form to represent at each step *the part of the root already found.*

It must be observed that *a represents so many tens with respect to the next figure of the root.*

Ex. Find the square root of 5,322,249.

$$
\begin{array}{r|l}
 & \dot{5}3\dot{2}2\dot{2}4\dot{9}\,(2307 \\
 & 4 \\
43) & 132 \\
 & 129 \\
4607) & 32249 \\
 & 32249
\end{array}
$$

212. If the square root of a number have decimal places, the number itself will have *twice* as many.

Thus, if .21 be the square root of some number, this number will be $(.21)^2 = .21 \times .21 = .0441$; and if .111 be the root, the number will be $(.111)^2 = .111 \times .111 = .012321$.

Therefore, the number of *decimal* places in every square decimal will be *even*, and the number of decimal places in the root will be *half* as many as in the given number itself.

Hence, if the given square number contain a decimal, and a dot be placed over the *units' figure*, and then over every *alternate* figure on *both* sides of it, the number of dots to the left of the decimal point will show the number of *integral* places in the root, and the number of dots to the right will show the number of *decimal* places.

Ex. Find the square roots of 41.2164 and 965.9664.

$$
\begin{array}{r|l}
 & 4\dot{1}.2\dot{1}6\dot{4}\,(6.42 \\
 & 36 \\
124) & 521 \\
 & 496 \\
1282) & 2564 \\
 & 2564
\end{array}
\qquad\qquad
\begin{array}{r|l}
 & \dot{9}6\dot{5}.9\dot{6}6\dot{4}\,(31.08 \\
 & 9 \\
61) & 65 \\
 & 61 \\
6208) & 49664 \\
 & 49664
\end{array}
$$

It is seen from the dotting that the root of the first example will have one integral and two decimal places, and that the root of the second example will have two integral and two decimal places.

213. If a number contain an *odd* number of decimal places, or if any number give a *remainder* when as many figures in the root have been obtained as the given number has periods, then its exact square root cannot be found. We may, however, approximate to its exact root as near as we please by annexing ciphers and continuing the operation.

Ex. Find the square roots of 3 and 357.357.

$$
\begin{array}{rl}
 & \dot{3}.(1.732\ldots\ldots \\
 & 1 \\
27) & 200 \\
 & 189 \\
343) & 1100 \\
 & 1029 \\
3462) & 7100 \\
 & 6924
\end{array}
\qquad\qquad
\begin{array}{rl}
 & \dot{3}5\dot{7}.3\dot{5}7\dot{0}(18.903\ldots\ldots \\
 & 1 \\
28) & 257 \\
 & 224 \\
369) & 3335 \\
 & 3321 \\
37803) & 147000 \\
 & 113409
\end{array}
$$

Exercise LXXIX.

Extract the square roots of:

1. 120,409; 4816.36; 1867.1041; 1435.6521; 64.128064.
2. 16,803.9369; 4.54499761; .24373969; .5687573056.
3. .9; 6.21; .43; .00852; 17; 129; 347.259.
4. 14,295.387; 2.5; 2000; .3; .03; 111.
5. .00111; .004; .005; 2; 5; 3.25; 8.6.
6. $\frac{1}{4}$; $\frac{16}{49}$; $\frac{100}{144}$; $\frac{169}{225}$; $\frac{289}{324}$; $\frac{400}{625}$.
7. $\frac{1}{2}$; $\frac{2}{3}$; $\frac{3}{4}$; $\frac{1}{32}$; $\frac{7}{128}$; $\frac{6}{125}$; $\frac{6}{7}$; $\frac{1}{12}$.

Cube Roots of Compound Expressions.

214. Since the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, the cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$.

It is required to find a method for extracting the cube root $a+b$ when $a^3+3a^2b+3ab^2+b^3$ is given:

(1) Find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

$$
\begin{array}{r|l}
 & a^3 + 3a^2b + 3ab^2 + b^3 \;\underline{|a+b} \\
3a^2 \qquad\qquad a^3 & \\
\underline{+3ab + b^2} & 3a^2b + 3ab^2 + b^3 \\
3a^2 + 3ab + b^2 & \underline{3a^2b + 3ab^2 + b^3}
\end{array}
$$

The first term a of the root is obviously the cube root of the first term a^3 of the given expression.

If a^3 be subtracted, the remainder is $3a^2b + 3ab^2 + b^3$; therefore, the second term b of the root is obtained by dividing the first term of this remainder by *three times the square of a.*

Also, since $3a^2b + 3ab^2 + b^3 = (3a^2 + 3ab + b^2)b$, the *complete divisor* is obtained by adding $3ab + b^2$ to the *trial-divisor* $3a^2$.

(2) Find the cube root of $8x^3 + 36x^2y + 54xy^2 + 27y^3$.

$$
\begin{array}{r|l}
 & 8x^3+36x^2y+54xy^2+27y^3\;\underline{|2x+3y} \\
12x^2 \qquad\qquad 8x^3 & \\
(6x+3y)3y = \underline{\qquad 18xy+9y^2} & 36x^2y+54xy^2+27y^3 \\
12x^2+18xy+9y^2 & \underline{36x^2y+54xy^2+27y^3}
\end{array}
$$

The cube root of the first term is $2x$, and this is therefore the first term of the root.

The second term of the root, $3y$, is obtained by dividing $36x^2y$ by $3(2x)^2 = 12x^2$, which corresponds to $3a^2$ in the typical form, and is completed by annexing to $12x^2$ the expression $\{3(2x) + 3y\}3y = 18xy + 9y^2$, which corresponds to $3ab + b^2$, in the typical form.

215. The same method may be applied to longer expressions by considering a in the typical form $3a^2 + 3ab + b^2$ to represent at each stage of the process *the part of the root already found.*

Thus, if the part of the root already found be $x + y$, then $3a^2$ of the typical form will be represented by $3(x + y)^2$; and if the third term of the root be $+z$, the $3ab + b^2$ will be represented by $3(x + y)z + z^2$. So that the complete divisor, $3a^2 + 3ab + b^2$, will be represented by $3(x + y)^2 + 3(x + y)z + z^2$.

Find the cube root of $x^6 - 3x^5 + 5x^5 - 3x - 1$.

$$\begin{array}{rrr|l}
 & & & x^2 - x - 1 \\
\hline
 & & & x^6 - 3x^5 + 5x^3 - 3x - 1 \\
 & 3x^4 & & x^6 \\
(3x^2 - x)(-x) = & -3x^3 + & x^2 & -3x^5 + 5x^3 \\
 & 3x^4 - 3x^3 + & x^2 & -3x^5 + 3x^4 - x^3 \\
\hline
 & & & -3x^4 + 6x^3 - 3x - 1 \\
3(x^2 - x)^2 = & 3x^4 - 6x^3 + 3x^2 & & \\
(3x^2 - 3x - 1)(-1) = & -3x^2 + 3x + 1 & & \\
\hline
 & 3x^4 - 6x^3 \quad + 3x + 1 & & -3x^4 + 6x^3 - 3x - 1 \\
\hline
\end{array}$$

The root is placed above the given expression for convenience of arrangement.

The first term of the root, x^2, is obtained by taking the cube root of the first term of the given expression; and the first trial-divisor, $3x^4$, is obtained by taking three times the square of this term of the root.

The first complete divisor is found by annexing to the trial-divisor $(3x^2 - x)(-x)$, which expression corresponds to $(3a + b)b$ in the typical form.

The part of the root already found (a) is now represented by $x^2 - x$; therefore, $3a^2$ is represented by $3(x^2 - x)^2 = 3x^4 - 6x^3 + 3x^2$, the second trial-divisor; and $(3a + b)b$ by $(3x^2 - 3x - 1)(-1)$; therefore, in the second complete divisor, $3a^2 + (3a + b)b$ is represented by

$$(3x^4 - 6x^3 + 3x^2) + (-3x^2 - 3x - 1) \times (-1) = 3x^4 - 6x^3 + 3x + 1.$$

EXERCISE LXXX.

Find the cube roots of:

1. $x^3 + 6x^2y + 12xy^2 + 8y^3$.
2. $a^3 - 9a^2 + 27a - 27$.
3. $x^3 + 12x^2 + 48x + 64$.
4. $x^6 - 3ax^5 + 5a^3x^3 - 3a^5x - a^6$.
5. $x^6 + 3x^5 + 6x^4 + 7x^3 + 6x^2 + 3x + 1$.
6. $1 - 9x + 39x^2 - 99x^3 + 156x^4 - 144x^5 + 64x^6$.
7. $a^6 - 6a^5 + 9a^4 + 4a^3 - 9a^2 - 6a - 1$.
8. $64x^6 + 192x^5 + 144x^4 - 32x^3 - 36x^2 + 12x - 1$.
9. $1 - 3x + 6x^2 - 10x^3 + 12x^4 - 12x^5 + 10x^6 - 6x^7 + 3x^8 - x^9$.

10. $a^6 + 9a^5b - 135a^3b^3 + 729ab^5 - 729b^6.$

11. $c^6 - 12bc^5 + 60b^2c^4 - 160b^3c^3 + 240b^4c^2 - 192b^5c + 64b^6.$

12. $3a^6 + 48a^5b + 60a^4b^2 - 80a^3b^3 - 90a^2b^4 + 108ab^5 - 27b^6.$

CUBE ROOTS OF ARITHMETICAL NUMBERS.

216. In extracting the cube root of a number expressed by figures, the first step is to mark it off into periods.

Since $1 = 1^3$, $1000 = 10^3$, $1,000,000 = 100^3$, and so on, it follows that the cube root of any number between 1 and 1000, that is, of any number which has *one*, *two*, or *three* figures, is a number of *one* figure; and that the cube root of any number between 1000 and 1,000,000, that is, of any number which has *four*, *five*, or *six* figures, is a number of *two* figures; and so on.

Hence, if a dot be placed over every *third* figure of a cube number, beginning with the *units' figure*, the number of dots will be equal to the number of figures in its cube root.

217. If the cube root of a number contain any decimal figures, the number itself will contain *three times* as many.

Thus, if .3 be the cube root of a number, the number is $.3 \times .3 \times .3 = .027$.

Hence, if the given cube number have decimal places, and a dot be placed over the *units' figure* and over every *third* figure on *both* sides of it, the number of dots to the *left* of the decimal point will show the number of *integral* figures in the root; and the number of dots to the *right* will show the number of *decimal* figures in the root.

If the given number be not a perfect cube, ciphers may be annexed, and a value of the root may be found as near to the *true* value as we please.

218. It is to be observed that if a denote the first term of the root, and b the second term, the *first complete divisor* is

$$3a^2 + 3ab + b^2,$$

and the *second trial-divisor* is $3(a+b)^2$, that is,

$$3a^2 + 6ab + 3b^2,$$

which may be obtained from the preceding complete divisor by adding to it its second term and twice its third term,

$$\begin{array}{r}
3a^2+3ab+\ \ b^2 \\
+3ab+2b^2 \\
\hline
3a^2+6ab+3b^2
\end{array}$$

a method which will very much shorten the work in long arithmetical examples.

219. Ex. Extract the cube root of 5 to five places of decimals.

$$\begin{array}{rr|l}
 & & 5.000(1.70997 \\
 & & 1 \\
\hline
3\times 10^2 = & 300 & 4\,000 \\
3(10\times 7) = & 210 & \\
7^2 = & 49 & \\
\cline{2-2}
 & 559 & 3\,913 \\
\hline
 & 259 & 87\,000\,000 \\
\cline{2-2}
3\times 1700^2 = & 8670000 & \\
3(1700\times 9) = & 45900 & \\
9^2 = & 81 & \\
\cline{2-2}
 & 8715981 & 78\,443\,829 \\
\cline{3-3}
 & 45981 & 8\,556\,1710 \\
\cline{2-2}
3\times 1709^2 = & 8762043 & 7\,885\,8387 \\
\cline{3-3}
 & & 670\,33230 \\
 & & 613\,34301 \\
\cline{3-3}
\end{array}$$

After the first two figures of the root are found, the next trial-divisor is obtained by bringing down the sum of the 210 and 49 obtained in completing the preceding divisor; then adding the three lines connected by the brace, and annexing two ciphers to the result.

The last two figures of the root are found by division. The rule in such cases is, that two less than the number of figures already obtained may be found without error by division, the divisor to be employed being three times the square of the part of the root already found.

EXERCISE LXXXI.

Find the cube roots of:

1. 274,625.
2. 110,592.
3. 262,144.
4. 884.736.
5. 109,215,352.
6. 1,481,544.
7. 1601.613.
8. 1,259,712.
9. 2.803221.
10. 7,077,888.
11. 12.812904.
12. 56.623104.
13. 33,076.161.
14. 102,503.232.
15. 820.025856.
16. 8653.002877.
17. 1.371330631.
18. 20,910.518875.
19. 91.398648466125.
20. 5.340104393239.
21. Find to four figures the cube roots of 2.5; .2; .01; 4; .4.

220. Since the fourth power is the square of the square, and the sixth power the square of the cube; the *fourth root* is the *square root* of the *square root*, and the *sixth root* is the *cube root* of the *square root*. In like manner, the eighth, ninth, twelfth..... roots may be found.

EXERCISE LXXXII.

Find the fourth roots of:

1. $81a^4 - 540a^3b + 1350a^2b^2 - 1500ab^3 + 625b^4$.
2. $1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8$.

Find the sixth roots of:

3. $64 - 192x + 240x^2 - 160x^3 + 60x^4 - 12x^5 + x^6$.
4. $729x^6 - 1458x^5 + 1215x^4 - 540x^3 + 135x^2 - 18x + 1$.

Find the eighth root of:

5. $1 - 8y + 28y^2 - 56y^3 + 70y^4 - 56y^5 + 28y^6 - 8y^7 + y^8$.

CHAPTER XIV.

QUADRATIC EQUATIONS.

221. An equation which contains the *square* of the unknown quantity, but no higher power, is called a **quadratic equation.**

222. If the equation contain the *square only*, it is called a **pure quadratic**; but if it contain the *first power also*, it is called an **affected quadratic.**

PURE QUADRATIC EQUATIONS.

Solve the equation $5x^2 - 48 = 2x^2$.

$$
\begin{aligned}
5x^2 - 48 &= 2x^2 \\
3x^2 &= 48 \\
x^2 &= 16 \\
\therefore x &= \pm 4
\end{aligned}
$$

It will be observed that there are *two* roots of equal value but of opposite signs; and there are only two, for if the square root of the equation, $x^2 = 16$, were written $\pm x = \pm 4$, there would be only two values of x; since the equation $-x = +4$ gives $x = -4$, and the equation $-x = -4$ gives $x = 4$.

Hence, to solve a pure quadratic,

Collect the unknown quantities on one side, and the known quantities on the other; divide by the co-efficient of the unknown quantity; and extract the square root of each side of the resulting equation.

Solve the equation $3x^2 - 15 = 0$.

$$
\begin{aligned}
3x^2 - 15 &= 0 \\
3x^2 &= 15 \\
x^2 &= 5 \\
\therefore x &= \pm\sqrt{5}
\end{aligned}
$$

It will be observed that the square root of 5 cannot be found exactly, but an approximate value of it to any assigned degree of accuracy may be found.

223. A root which is indicated, but which can be found only approximately, is called a **Surd.**

Solve the equation $3x^2+15=0$.

$$\begin{aligned} 3x^2+15&=0\\ 3x^2&=-15\\ x^2&=-5\\ \therefore x&=\pm\sqrt{-5} \end{aligned}$$

It will be observed that the square root of -5 cannot be found even approximately; for the square of any number, positive or negative, is positive.

224. A root which is indicated, but which cannot be found exactly or approximately, is **imaginary.** § 206.

Exercise LXXXIII.

Solve:

1. $x^2-3=46$.
2. $2(x^2-1)-3(x^2+1)+14=0$.
3. $\dfrac{x^2-5}{3}+\dfrac{2x^2+1}{6}=\dfrac{1}{2}$.
4. $\dfrac{3}{1+x}+\dfrac{3}{1-x}=8$.
5. $\dfrac{3}{4x^2}-\dfrac{1}{6x^2}=\dfrac{7}{3}$.
6. $5x^2-9=2x^2+24$.
7. $(x+2)^2=4x+5$.
8. $\dfrac{x^2}{5}-\dfrac{x^2-10}{15}=7-\dfrac{50+x^2}{25}$.
9. $\dfrac{3x^2-27}{x^2+3}+\dfrac{90+4x^2}{x^2+9}=7$.
10. $8x+\dfrac{7}{x}=\dfrac{65x}{7}$.
11. $\dfrac{4x^2+5}{10}-\dfrac{2x^2-5}{15}=\dfrac{7x^2-25}{20}$.
12. $\dfrac{10x^2+17}{18}-\dfrac{12x^2+2}{11x^2-8}=\dfrac{5x^2-4}{9}$.
13. $\dfrac{14x^2+16}{21}-\dfrac{2x^2+8}{8x^2-11}=\dfrac{2x^2}{3}$.
14. $x^2+bx+a=bx(1-bx)$.
15. $mx^2+n=q$.
16. $x^2-ax+b=ax(x-1)$.

Affected Quadratic Equations.

225. Since $(ax \pm b)^2 = a^2x^2 \pm 2abx + b^2$, it is evident that the expression $a^2x^2 \pm 2abx$ lacks only the *third term*, b^2, of being a complete square.

It will be seen that this third term is *the square of the quotient obtained from dividing the second term by twice the square root of the first term.*

226. Every affected quadratic may be made to assume the form of $a^2x^2 \pm 2abx = c$.

The first step in the solution of such an equation is to *complete the square*; that is, to add to each side *the square of the quotient obtained from dividing the second term by twice the square root of the first term.*

The second step is to *extract the square root* of each side of the resulting equation.

The third and last step is to *reduce* the resulting simple equation.

(1) Solve the equation $16x^2 + 5x - 3 = 7x^2 - x + 45$.

$$16x^2 + 5x - 3 = 7x^2 - x + 45.$$

Simplify, $9x^2 + 6x = 48.$

Complete the square, $9x^2 + 6x + 1 = 49.$

Extract the root, $3x + 1 = \pm 7.$

Reduce, $3x = -1 + 7 \text{ or } -1 - 7,$

$$3x = 6 \text{ or } -8,$$

$$\therefore\ x = 2 \text{ or } -2\tfrac{2}{3}.$$

Verify by substituting 2 for x in the equation

$$16x^2 + 5x - 3 = 7x^2 - x + 45,$$

$$16(2)^2 + 5(2) - 3 = 7(2)^2 - (2) + 45,$$

$$64 + 10 - 3 = 28 - 2 + 45,$$

$$71 = 71.$$

Verify by substituting $-2\frac{2}{3}$ for x in the equation

$$16x^2 + 5x - 3 = 7x^2 - x + 45,$$
$$16\left(-\tfrac{8}{3}\right)^2 + 5\left(-\tfrac{8}{3}\right) - 3 = 7\left(-\tfrac{8}{3}\right)^2 - \left(-\tfrac{8}{3}\right) + 45;$$
$$\tfrac{1024}{9} - \tfrac{40}{3} - 3 = \tfrac{448}{9} + \tfrac{8}{3} + 45,$$
$$1024 - 120 - 27 = 448 + 24 + 405,$$
$$877 = 877.$$

(2) Solve the equation $3x^2 - 4x = 32$.

Since the exact root of 3, the coefficient of x^2, cannot be found, it is necessary to multiply or divide each term of the equation by 3 to make the coefficient of x^2 a *square number*.

Multiply by 3, $9x^2 - 12x = 96$.
Complete the square, $9x^2 - 12x + 4 = 100$.
Extract the root, $3x - 2 = \pm 10$.
Reduce, $3x = 2 + 10$ or $2 - 10$;
$3x = 12$ or -8.
$\therefore x = 4$ or $-2\frac{2}{3}$.

Or, divide by 3, $x^2 - \dfrac{4x}{3} = \dfrac{32}{3}$.

Complete the square, $x^2 - \dfrac{4x}{3} + \dfrac{4}{9} = \dfrac{32}{3} + \dfrac{4}{9} = \dfrac{100}{9}$.

Extract the root, $x - \dfrac{2}{3} = \pm \dfrac{10}{3}$.

$$\therefore x = \frac{2 \pm 10}{3},$$
$$= 4 \text{ or } -2\tfrac{2}{3}.$$

Verify by substituting 4 for x in the original equation,

$$48 - 16 = 32,$$
$$32 = 32.$$

Verify by substituting $-2\frac{2}{3}$ for x in the original equation.

$$21\tfrac{1}{3} - \left(-10\tfrac{2}{3}\right) = 32,$$
$$32 = 32.$$

(3) Solve the equation $-3x^2+5x=-2$.

Since the *even* root of a *negative* number is impossible, it is necessary to change the sign of each term. The resulting equation is,

$$3x^2-5x=2.$$

Multiply by 3, $\quad 9x^2-15x=6.$

Complete the square, $\quad 9x^2-15x+\frac{25}{4}=\frac{49}{4}.$

Extract the root, $\quad 3x-\frac{5}{2}=\pm\frac{7}{2}.$

Reduce, $\quad 3x=\frac{5\pm 7}{2},$

$$3x=6 \text{ or } -1.$$

$$\therefore\ x=2 \text{ or } -\tfrac{1}{3}.$$

Or, divide by 3, $\quad x^2-\frac{5x}{3}=\frac{2}{3}.$

Complete the square, $\quad x^2-\frac{5x}{3}+\frac{25}{36}=\frac{49}{36}.$

Extract the root, $\quad x-\frac{5}{6}=\pm\frac{7}{6}.$

$$\therefore\ x=\frac{5\pm 7}{6},$$

$$=2 \text{ or } -\tfrac{1}{3}.$$

If the equation $3x^2-5x=2$ be multiplied by *four times the coefficient of* x^2, fractions will be avoided:

$$36x^2-60x=24.$$

Complete the square, $\quad 36x^2-60x+25=49.$

Extract the root, $\quad 6x-5=\pm 7,$

$$6x=5\pm 7,$$

$$6x=12 \text{ or } -2.$$

$$\therefore\ x=2 \text{ or } -\tfrac{1}{3}.$$

It will be observed that the number added to complete the square by this last method is *the square of the coefficient of* x in the original equation $3x^2-5x=2$.

(4) Solve the equation $\frac{3}{5-x} - \frac{1}{2x-5} = 2.$

Simplify (as in simple equations),

$$4x^2 - 23x = -30.$$

Multiply by four times the coefficient of x^2, and add to each side the square of the coefficient of x,

$$64x^2 - (\quad) + (23)^2 = 529 - 480 = 49.$$

Extract the root, $\quad 8x - 23 = \pm 7.$

Reduce, $\quad 8x = 23 \pm 7;$

$$8x = 30 \text{ or } 16.$$

$$\therefore x = 3\tfrac{3}{4} \text{ or } 2.$$

If a trinomial be a perfect square, its root is found by taking the roots of the *first* and *third* terms and connecting them by the *sign* of the middle term. It is not necessary, therefore, in completing the square, to write the middle term, but its place may be indicated as in this example.

(5) Solve the equation $72x^2 - 30x = -7.$

Since $72 = 2^3 \times 3^2$, if the equation be multiplied by 2, the coefficient of x^2 in the resulting equation, $144x^2 - 60x = -14$, will be a square number, and the term required to complete the square will be $(\frac{60}{24})^2 = (\frac{5}{2})^2 = \frac{25}{4}$. Hence, if the original equation be multiplied by 4×2, the coefficient of x^2 in the result will be a square number, and fractions will be avoided in the work.

Multiply the given equation by 8,

$$576x^2 - 240x = -56.$$

Complete the square, $576x^2 - (\quad) + 25 = -31.$

Extract the root, $\quad 24x - 5 = \pm\sqrt{-31}.$

Reduce, $\quad 24x = 5 \pm \sqrt{-31}.$

$$\therefore x = \tfrac{1}{24}(5 \pm \sqrt{-31}).$$

NOTE. In solving the following equations, care must be taken to select the method best adapted to the example under consideration.

EXERCISE LXXXIV.

Solve:

1. $x^2 + 4x = 12.$	4. $x^2 - 7x = 8.$	7. $x^2 - x = 6.$
2. $x^2 - 6x = 16.$	5. $3x^2 - 4x = 7.$	8. $5x^2 - 3x = 2.$
3. $x^2 - 12x + 6 = \frac{1}{4}.$	6. $12x^2 + x - 1 = 0.$	9. $2x^2 - 27x = 14.$

10. $x^2 - \frac{2x}{3} + \frac{1}{12} = 0.$

11. $\frac{x^2}{2} - \frac{x}{3} = 2(x+2).$

12. $\frac{3x}{4} + \frac{4}{3x} = \frac{13}{6}.$

13. $\frac{x+1}{x+4} = \frac{2x-1}{x+6}.$

14. $\frac{x}{x+1} - \frac{x+3}{2(x+4)} = -\frac{1}{18}.$

15. $\frac{2}{x-1} = \frac{3}{x-2} + \frac{2}{x-4}.$

16. $5x(x-3) - 2(x^2-6) = (x+3)(x+4).$

17. $\frac{3x}{2(x+1)} - \frac{5}{8} = \frac{3x^2}{x^2-1} - \frac{23}{4(x-1)}.$

18. $(x-2)(x-4) - 2(x-1)(x-3) = 0.$

19. $\frac{1}{7}(x-4) - \frac{2}{5}(x-2) = \frac{1}{x}(2x+3).$

20. $\frac{2}{5}(3x^2 - x - 5) - \frac{1}{3}(x^2-1) = 2(x-2)^2.$

21. $\frac{2x}{15} + \frac{3x-50}{3(10+x)} = \frac{12x+70}{190}.$

22. $\frac{x}{x^2-1} = \frac{15-7x}{8(1-x)}.$

23. $\frac{2x-1}{x-1} + \frac{1}{6} = \frac{2x-3}{x-2}.$

24. $\frac{x+2}{x-1} - \frac{4-x}{2x} = \frac{7}{3}.$

25. $x - \frac{14x-9}{8x-3} = \frac{x^2-3}{x+1}.$

26. $1 - \frac{x+5}{2x+1} = \frac{x-6}{x-2}.$

27. $\frac{x}{7-x} + \frac{7-x}{x} = 2\frac{9}{10}.$

28. $\frac{2x+3}{2(2x-1)} - \frac{7-x}{2(x+1)} = \frac{7-3x}{4-3x}.$

29. $\frac{12x^3 - 11x^2 + 10x - 78}{8x^2 - 7x + 6} = 1\frac{1}{2}x - \frac{1}{2}.$

30. $\frac{6}{x-1} - \frac{18}{x+5} = \frac{7}{x+1} - \frac{8}{x-5}.$

227. *Literal quadratic equations* are solved as follows:

(1) Solve the equation $ax^2 + bx = c$.

Multiply the equation by $4a$ and add the square of b,

$$4a^2x^2 + (\ \ ) + b^2 = 4ac + b^2.$$

Extract the root, $2ax + b = \pm\sqrt{4ac + b^2}.$

Reduce, $2ax = -b \pm\sqrt{4ac + b^2}.$

$$\therefore\ x = \frac{-b \pm\sqrt{4ac + b^2}}{2a}.$$

(2) Solve the equation $adx - acx^2 = bcx - bd$.

Transpose bcx and change the signs,

$$acx^2 + bcx - adx = bd.$$

Express the left member in *two terms*,

$$acx^2 + (bc - ad)\,x = bd.$$

Multiply by $4ac$,

$$4a^2c^2x^2 + 4ac\,(bc - ad)\,x = 4abcd.$$

Complete the square,

$$4a^2c^2x^2 + (\ \ ) + (bc - ad)^2 = b^2c^2 + 2abcd + a^2d^2.$$

Extract the root, $2acx + (bc - ad) = \pm(bc + ad).$

Reduce, $2acx = -(bc - ad) \pm (bc + ad)$

$$= 2ad \text{ or } -2bc.$$

$$\therefore\ x = \frac{d}{c} \text{ or } -\frac{b}{a}.$$

(3) Solve the equation $px^2 - px + qx^2 + qx = \dfrac{pq}{p+q}$.

Express the left member in *two terms*,

$$(p + q)\,x^2 - (p - q)\,x = \frac{pq}{p+q}.$$

Multiply by four times the coefficient of x^2,

$$4(p + q)^2x^2 - 4(p^2 - q^2)\,x = 4pq.$$

Complete the square,

$$4(p + q)^2x^2 - (\ \ ) + (p - q)^2 = p^2 + 2pq + q^2.$$

Extract the root, $2(p + q)\,x - (p - q) = \pm(p + q).$

Reduce, $2(p + q)\,x = (p - q) \pm (p + q),$

$$= 2p \text{ or } -2q.$$

$$\therefore\ x = \frac{p}{p+q} \text{ or } -\frac{q}{p+q}.$$

NOTE. The left-hand member of the equation when simplified must be expressed in *two terms, simple or compound*, one term containing x^2, and the other term containing x.

Exercise LXXXV.

Solve:

1. $x^2+2ax=a^2$.

2. $x^2=4ax+7a^2$.

3. $x^2=\frac{7m^2}{4}-3mx$.

4. $x^2-\frac{5nx}{2}-\frac{3n^2}{2}=0$.

5. $\frac{a^2}{(x+a)^2}=\frac{b^2}{(x-a)^2}$.

6. $cx=ax^2+bx^2-\frac{ac}{a+b}$.

7. $\frac{a^2x^2}{b^2}+\frac{b^2}{c^2}=\frac{2ax}{c}$.

8. $(a^2+1)x=ax^2+a$.

9. $\frac{a}{x-a}+\frac{b}{x-b}=\frac{2c}{x-c}$.

10. $\frac{1}{a+b+x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{x}$.

11. $\frac{1}{a-x}-\frac{1}{a+x}=\frac{3+x^2}{a^2-x^2}$.

12. $\frac{x^2+2ab(a^2+b^2)}{a^2+b^2}=2x$.

13. $\frac{(2x-a)^2}{2x-a+2b}=b$.

14. $x^2+ax=a+x$.

15. $x^2+ax=bx+ab$.

16. $\frac{x}{a}+\frac{a}{x}=\frac{x}{b}+\frac{b}{x}$.

17. $\frac{1}{x}+\frac{1}{x+b}=\frac{1}{a}+\frac{1}{a+b}$.

18. $\frac{a}{3}+\frac{5x}{4}-\frac{x^2}{3a}=0$.

19. $\frac{x+3}{x-3}=a+\frac{x-3}{x+3}$.

20. $mx^2-1=\frac{x(m^3-n^2)}{mn}$.

21. $(ax-b)(bx-a)=c^2$.

22. $\frac{ax+b}{bx+a}=\frac{mx+n}{nx+m}$.

23. $\frac{m}{m+x}+\frac{m}{m-x}=c$.

24. $\frac{(a-1)^2x^2+2(3a-1)x}{4a-1}=1$.

25. $\frac{(a^2-b^2)(x^2+1)}{a^2+b^2}=2x$.

26. $\frac{x^2-4mnx}{(m+n)^2}=(m-n)^2$.

27. $x^2+\frac{a-b}{ab^2}=\frac{14a^2-5ab-10b^2}{18a^2b^2}+\frac{(2a-3b)x}{2ab}$.

28. $abx^2+\dfrac{b^2x}{c}=\dfrac{6a^2+ab-2b^2}{c^2}-\dfrac{3a^2x}{c}.$

29. $\dfrac{x^2}{3m-2a}-\dfrac{m^2-4a^2}{4a-6m}=\dfrac{x}{2}.$

30. $6x+\dfrac{(a+b)^2}{x}=5(a-b)+\dfrac{25ab}{6x}.$

31. $\frac{8}{9}(x^2+a^2+ab)=\frac{1}{9}x(20a+4b).$

32. $x^2-(b-a)c=ax-bx+cx.$

33. $x^2-2mx=(n-p+m)(n-p-m).$

34. $x^2-(m+n)x=\frac{1}{4}(p+q+m+n)(p+q-m-n).$

35. $mnx^2-(m+n)(mn+1)x+(m+n)^2=0.$

36. $\dfrac{2b-x-2a}{bx}+\dfrac{4b-7a}{ax-bx}=\dfrac{x-4a}{ab-b^2}.$

37. $2x^2(a^2-b^2)-(3a^2+b^2)(x-1)=(3b^2+a^2)(x+1).$

38. $\dfrac{a-2b-x}{a^2-4b^2}-\dfrac{5b-x}{ax+2bx}+\dfrac{2a-x-19b}{2bx-ax}=0.$

39. $\dfrac{x+13a+3b}{5a-3b-x}-1=\dfrac{a-2b}{x+2b}.$

40. $\dfrac{x+3b}{8a^2-12ab}-\dfrac{3b}{9b^2-4a^2}-\dfrac{a+3b}{(2a+3b)(x-3b)}=0.$

41. $nx^2+px-px^2-mx+m-n=0.$

42. $(a+b+c)x^2-(2a+b+c)x+a=0.$

43. $(ax-b)(c-d)=(a-b)(cx-d)x.$

44. $\dfrac{2x+1}{b}-\dfrac{1}{x}\left(\dfrac{1}{b}-\dfrac{2}{a}\right)=\dfrac{3x+1}{a}.$

45. $\dfrac{1}{2x^2+x-1}+\dfrac{1}{2x^2-3x+1}=\dfrac{a}{2bx-b}-\dfrac{2bx+b}{ax^2-a}.$

228. An affected quadratic may be reduced to the form $x^2+px+q=0$, in which p and q represent *any* numbers, positive or negative, integral or fractional.

Ex. Solve: $x^2+px+q=0.$

$$4x^2+(\ )+p^2=p^2-4q,$$

$$2x+p=\pm\sqrt{p^2-4q},$$

$$\therefore\ x=-\frac{p}{2}\pm\frac{1}{2}\sqrt{p^2-4q}.$$

By this formula, the values of x in an equation of the form $x^2+px+q=0$, may be written at once. Thus, take the equation

$$3x^2-5x+2=0.$$

Divide by 3, $x^2-\frac{5}{3}x+\frac{2}{3}=0.$

Here, $p=-\frac{5}{3}$, and $q=\frac{2}{3}$.

$$\therefore x=\tfrac{5}{6}\pm\tfrac{1}{2}\sqrt{\tfrac{25}{9}-\tfrac{8}{3}},$$

$$=\tfrac{5}{6}\pm\tfrac{1}{6},$$

$$=1 \text{ or } \tfrac{2}{3}.$$

229. A quadratic which has been reduced to its simplest form, and has all its terms written on one side, may often have that side resolved *by inspection* into factors.

In this case, the roots are seen at once without completing the square.

(1) Solve $x^2+7x-60=0$.

Since $x^2+7x-60=(x+12)(x-5)$,
the equation $x^2+7x-60=0$
may be written $(x+12)(x-5)=0$.

It will be observed that if *either* of the factors $x+12$ or $x-5$ is 0, the *product of the two factors* is 0, and the equation is satisfied.

Hence, $x+12=0$ and $x-5=0$.

$$\therefore x=-12, \text{ and } x=5.$$

(2) Solve $x^2+7x=0$.

The equation $\qquad x^2+7x=0$
becomes $\qquad x(x+7)=0,$
and is satisfied $\qquad$ if $x=0$, or if $x+7=0$.

$\therefore$ the roots are 0 and -7.

It will be observed that this method is easily applied to an equation *all* the terms of which contain x.

(3) Solve $2x^3-x^2-6x=0$.

The equation $\qquad 2x^3-x^2-6x=0$
becomes $\qquad x(2x^2-x-6)=0,$
and is satisfied $\qquad$ if $x=0$, or if $2x^2-x-6=0$.

By solving $2x^2-x-6=0$ the two roots 2 and $-\frac{3}{2}$ are found.

$\therefore$ the equation has *three* roots, 0, 2, $-\frac{3}{2}$.

(4) Solve $x^3+x^2-4x-4=0$.

The equation $\qquad x^3+x^2-4x-4=0$
becomes $\qquad x^2(x+1)-4(x+1)=0,$
$$(x^2-4)(x+1)=0.$$

$\therefore$ the roots of the equation are -1, 2, -2.

(5) Solve $x^3-2x^2-11x+12=0$.

Since $\qquad \dfrac{x^3-2x^2-11x+12}{x-1}=x^2-x-12,$
the equation $\qquad x^3-2x^2-11x+12=0$
may be written $\qquad (x-1)(x^2-x-12)=0.$

The three roots are found to be 1, -3, 4.

An equation which cannot be resolved into factors by inspection may sometimes be solved by *guessing* at a root and reducing by division. In this case, if a denote the root, the given equation (all the terms of the equation being written on one side), may be divided by $x-a$.

EXERCISE LXXXVI.

Find the roots of:

1. $(x+1)(x-2)(x^2+x-2)=0.$
2. $(x^2-3x+2)(x^2-x-12)=0.$
3. $(x+1)(x-2)(x+3)=-6.$
4. $2x^3+4x^2-70x=0.$
5. $(x^2-x-6)(x^2-x-20)=0.$
6. $x(x+1)(x+2)=(a+2)(a+1)a.$
7. $x^3-x^2-x+1=0.$
8. $8x^3-1=0.$
9. $8x^3+1=0.$
10. $x^6-1=0.$
11. $x(x-a)(x^2-b^2)=0.$
12. $n(x^3+1)+x+1=0.$

230. If r and r' represent two values of x, then

$$x-r=0,$$

and
$$x-r'=0,$$

$$\therefore (x-r)(x-r')=0.$$

This is a quadratic equation, as may be seen by performing the indicated multiplication.

Now r and r' are roots of this equation; for, if either r or r' be written for x, one of the factors, $x-r$, $x-r'$, is equal to 0, and the equation is satisfied. Also r and r' are the *only* roots, for no value of x, except r and r', can make either of these factors equal to 0.

Since r and r' may represent the values of x in any quadratic equation, it follows that every quadratic equation has *two* roots, and *only two.*

Again, if r, r', r'', represent three values of x,
then,
$$(x-r)(x-r')(x-r'')=0.$$

This is a *cubic* equation, as may be seen by performing the indicated multiplication. Hence, it may be inferred that a *cubic* equation has *three* roots, and *only three;* and so, for any equation, that the *number* of roots is equal to the *degree* of the equation.

It may also be inferred that if r be a root of an equation, $x-r$ *will be a factor of the equation* when the equation is written with all its terms on one side.

If r and r' represent the roots of the general quadratic equation,
$$x^2+px+q=0.$$
This equation may be written $(x-r)(x-r')=0,$
or, $$x^2-(r+r')x+rr'=0.$$
A form which shows that
the *sum* of the roots $=-p,$
and the *product* of the roots $=q.$

231. It will be seen from § 230 that an equation may be formed if its roots be known.

If the roots of an equation be -1 and $\frac{1}{4}$,
the equation will be $(x+1)(x-\frac{1}{4})=0,$
or, $x^2+\dfrac{3x}{4}-\dfrac{1}{4}=0,$
or, by multiplying by 4, $4x^2+3x-1=0.$

If the roots of an equation be 0, 1, 5,
the equation will be $(x-0)(x-1)(x-5)=0;$
that is, $x(x-1)(x-5)=0,$
or, $x^3-6x^2+5x=0.$

If x occur in *every* term, the equation will be satisfied by putting $x=0$, and may be reduced to an equation of the next lower degree by dividing every term by x.

232. By considering the roots of $x^2+px+q=0$,

namely, $$r=-\frac{p}{2}+\frac{1}{2}\sqrt{p^2-4q},$$

and $$r'=-\frac{p}{2}-\frac{1}{2}\sqrt{p^2-4q},$$

it will be seen that the *character of the roots* of an equation may be determined without solving it:

I. As the two roots have the same expression, $\sqrt{p^2-4q}$, *both* roots will be *real*, or *both* will be *imaginary*.

If both be real, *both* will be *rational* or *both surds*, according as p^2-4q *is* or *is not* a perfect square.

II. When p^2 *is greater than* $4q$, the two roots will be *real*, for then the expression $p^2 - 4q$ is *positive*, and therefore $\sqrt{p^2 - 4q}$ can be found exactly or approximately.

Since also its value in one root is to be added to $-\frac{p}{2}$, and in the other to be subtracted from $-\frac{p}{2}$, the two roots will be *different in value.*

III. When p^2 *is equal to* $4q$, the roots will be *equal in value.*

IV. When p^2 *is less than* $4q$, the roots will be *imaginary*, for then the expression $p^2 - 4q$ will be negative, and therefore $\sqrt{p^2 - 4q}$ represents the *even* root of a *negative* number, and is imaginary.

V. If $q\ (= r \times r')$ be *positive*, the roots, if real, will have the *same sign*, but opposite to that of p (since $r + r' = -p$).

But if q be *negative*, the roots will have *opposite signs.*

233. Determine by inspection the character of the roots of:

(1) $x^2 - 5x + 6 = 0$.

In this equation p is -5, and q is 6.

$$\therefore \sqrt{p^2 - 4q} = \sqrt{25 - 24} = 1.$$

$\therefore$ the roots will be *rational*, and *both positive.*

(2) $x^2 + 3x + 1 = 0$.

In this equation, p is 3, and q is 1.

$$\therefore \sqrt{p^2 - 4q} = \sqrt{9 - 4} = \sqrt{5}.$$

$\therefore$ the roots will be *surds*, and *both negative.*

(3) $x^2 + 3x + 4 = 0$.

In this equation p is 3, and q is 4.

$$\therefore \sqrt{p^2 - 4q} = \sqrt{9 - 16} = \sqrt{-7}.$$

$\therefore$ the roots will be *impossible.*

Exercise LXXXVII.

Form the equations whose roots are:

1. $2, 1.$
2. $7, -3.$
3. $\frac{1}{2}, \frac{1}{3}.$
4. $\frac{2}{3}, -\frac{3}{2}.$
5. $-5, -\frac{1}{2}.$
6. $-\frac{7}{9}, \frac{9}{7}.$
7. $3, -3, \frac{3}{4}, -\frac{3}{4}.$
8. $0, 1, 2, 3.$
9. $0, -\frac{1}{2}, \frac{3}{2}, -1.$
10. $a-2b, 3a+2b.$
11. $2a-b, b-3a.$
12. $a(a+1), 1-a.$

Determine by inspection the character of the roots of:

13. $x^2-7x+12=0.$
14. $x^2-7x-30=0.$
15. $x^2+4x-5=0.$
16. $5x^2+8=0.$
17. $x^2+4x+1=0.$
18. $x^2-2x+9=0.$
19. $3x^2-4x-4=0.$
20. $x^2+4x+4=0.$

234. It is often useful to determine the maximum or minimum value of a given *quadratic expression*.

(1) Find the maximum or minimum value of $1+x-x^2$.

Let $\quad 1+x-x^2=m;$

then, $\quad x^2-x=1-m,$

and $\quad 4x^2-(\ )+1=5-4m,$

$$2x-1=\pm\sqrt{5-4m}.$$

$$\therefore x=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{5-4m}.$$

Now, for all possible values of x, $5-4m$ cannot be negative; that is, m cannot be *greater* than $\frac{5}{4}$; and for this value x is $\frac{1}{2}$. Therefore, $\frac{5}{4}$ is the *maximum* value of the given expression.

(2) Find the maximum or minimum value of x^2+3x+4.

Let $\quad x^2+3x+4=m;$

then, $\quad x^2+3x=m-4,$

and, $\quad 4x^2+(\ )+9=4m-7,$

$$2x+3=\pm\sqrt{4m-7}.$$

$$x=-\tfrac{3}{2}\pm\tfrac{1}{2}\sqrt{4m-7}.$$

For all possible values of x, $4m-7$ cannot be negative; that is, m cannot be *less* than $\frac{7}{4}$; and for this value $x=-\frac{3}{2}$. Therefore, $\frac{7}{4}$ is the *minimum* value of the given expression.

EXERCISE LXXXVIII.

Find the maximum or minimum value (and determine which) of:

1. $4+6x-x^2$.
2. $\dfrac{(x+a)^2}{x}$.
3. $\dfrac{x^2+1}{x}$.
4. $(a-x)(x-b)$.
5. $\dfrac{x}{1+x^2}$.
6. $x^2+8x+20$.
7. x^2-2x+9.
8. $\dfrac{x^2}{(x+a)(x-b)}$.
9. $\dfrac{x}{a+x^2}$.
10. Divide a line 20 in. long into two parts so that the sum of the squares on these two parts may be the least possible.
11. Divide a line 20 in. long into two parts so that the rectangle contained by the parts may be the greatest possible.
12. Find the fraction which has the greatest excess over its square.

235. Two other cases of the solution of equations *by completing the square* should be noticed.

I. When *any* two powers of x are involved, *one of which is the square of the other.*

II. When the *addition of a number* to an equation of the fourth degree will *make both sides complete squares.*

(1) Solve $8x^6+63x^3=8$.

In this equation the exponent 6 is the double of 3, hence x^6 is the square of x^3.

$$8x^6+63x^3=8,$$
$$256x^6+(\ )+(63)^2=4225,$$
$$16x^3+63=\pm 65,$$
$$16x^3=2, \text{ or } -128,$$
$$x^3=\tfrac{1}{8}, \text{ or } -8.$$

By taking cube root, $\quad x=\tfrac{1}{2}, \text{ or } -2.$

The other roots of the equation are found by finding the remaining roots of the equations, $x^3 = \frac{1}{8}$, and $x^3 = -8$.

Since, $x^3 = \frac{1}{8}$, $\therefore 8x^3 - 1 = 0$

Now, by § 230,

$$8x^3 - 1 = (2x - 1)(4x^2 + 2x + 1)$$

$$\therefore (2x - 1)(4x^2 + 2x + 1) = 0$$

and is satisfied if $4x^2 + 2x + 1 = 0$

as well as if $2x - 1 = 0$

The solution of $4x^2 + 2x + 1 = 0$

gives $x = \frac{1}{4}(-1 \pm \sqrt{-3})$.

Since, $x^3 = -8$, $\therefore x^3 + 8 = 0$

Now, by § 230,

$$x^3 + 8 = (x + 2)(x^2 - 2x + 4)$$

$$\therefore (x + 2)(x^2 - 2x + 4) = 0$$

and is satisfied if $x^2 - 2x - 4 = 0$

as well as if $x + 2 = 0$

The solution of $x^2 - 2x + 4 = 0$

gives $x = 1 \pm \sqrt{-3}$.

$\therefore$ the roots are $\frac{1}{2}$, -2, $1 \pm \sqrt{-3}$, $\frac{1}{4}(-1 \pm \sqrt{-3})$.

(2) Solve $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$.

Take the square root of the left side.

$$\begin{array}{r|l}
 & x^4 - 10x^3 + 35x^2 - 50x + 24 \;\big|\; x^2 - 5x + 5 \\
 & x^4 \\
2x^2 - 5x & -10x^3 + 35x^2 \\
 & -10x^3 - 25x^2 \\
\hline
2x^2 - 10x + 5 & 10x^2 - 50x + 24 \\
 & 10x^2 - 50x + 25 \\
\hline
 & -\ 1
\end{array}$$

It is now seen that if 1 were added, the square would be complete and the equation would be

$$x^4 - 10x^3 + 35x^2 - 50x + 25 = 1.$$

Extract the square root, and the result is,

$$x^2 - 5x + 5 = \pm 1.$$

That is,

$$x^2 - 5x = -4, \text{ or } -6,$$

$$4x^2 - (\ ) + 25 = 9, \text{ or } 1,$$

$$2x - 5 = \pm 3, \text{ or } \pm 1,$$

$$2x = 8,\ 2,\ 6, \text{ or } 4.$$

$$\therefore x = 4,\ 1,\ 3, \text{ or } 2.$$

Exercise LXXXIX.

Find the *possible* roots of:

1. $x^6 + 7x^3 = 8.$
2. $x^4 - 5x^2 + 4 = 0.$
3. $37x^2 - 9 = 4x^4.$
4. $16x^8 = 17x^4 - 1.$
5. $32x^{10} - 33x^5 + 1 = 0.$
6. $(x^2-2)^2 = \frac{1}{4}(x^2+12)^2.$
7. $x^{4n} - \dfrac{5x^{2n}}{3} - \dfrac{25}{12} = 0.$
8. $(x^2-9)^2 = 3 + 11(x^2-2).$
9. $x^6 + 14x^3 + 24 = 0.$
10. $19x^4 + 216x^7 = x.$
11. $x^8 + 22x^4 + 21 = 0.$
12. $x^{2m} + 3x^m - 4 = 0.$
13. $4x^4 - 20x^3 + 23x^2 + 5x = 6.$
14. $\dfrac{1}{x^{2n}} + \dfrac{3}{x^n} - 20 = 0.$
15. $x^4 - 4x^3 - 10x^2 + 28x - 15 = 0.$
16. $x^4 - 2x^3 - 13x^2 + 14x = -24.$
17. $108x^4 = 20x(9x^2-1) - 51x^2 + 7.$
18. $(x^2-1)(x^2-2) + (x^2-3)(x^2-4) = x^4 + 5.$

Problems Involving Quadratics.

236. Problems which involve quadratic equations have apparently *two* solutions, as a quadratic has *two* roots. Sometimes both will be solutions; but generally one only will be a solution, and the other be inconsistent with the conditions of the problem. No difficulty will be found in selecting the result which belongs to the problem, and sometimes a change may be made in the statement of a problem so as to form a new problem corresponding to the solution which was inapplicable to the original problem.

(1) The sum of the squares of two consecutive numbers is 481. Find the numbers.

Let $x =$ one number,

and $x + 1 =$ the other.

Then $x^2 + (x + 1)^2 = 481,$

or $2x^2 + 2x + 1 = 481.$

The solution of which gives, $x = 15$, or -16.

The positive root 15 gives for the numbers, 15 and 16.

The negative root -16 is inapplicable to the problem, as *consecutive numbers* are understood to be integers which follow one another in the common scale, 1, 2, 3, 4.....

(2) What is the price of eggs per dozen when 2 more in a shilling's worth lowers the price 1 penny per dozen?

Let $x =$ number of eggs for a shilling.

Then, $\frac{1}{x} =$ cost of 1 egg in shillings,

and $\frac{12}{x} =$ cost of 1 dozen in shillings.

But, if $x + 2 =$ number of eggs for a shilling,

$\frac{12}{x+2} =$ cost of 1 dozen in shillings.

$\therefore \frac{12}{x} - \frac{12}{x+2} = \frac{1}{12}$ (1 penny being $\frac{1}{12}$ of a shilling).

The solution of which gives $x = 16$, or -18.

And, if 16 eggs cost a shilling, 1 dozen will cost $\frac{12}{16}$ of a shilling, or 9 pence.

Therefore, the price of the eggs is 9 pence per dozen.

If the problem be changed so as to read: What is the price of eggs per dozen when two *less* in a shilling's worth *raises* the price 1 penny per dozen? the algebraic statement will be

$$\frac{12}{x-2} - \frac{12}{x} = \frac{1}{12}.$$

The solution of which gives $x = 18$, or -16.

Hence, the number 18, which had a negative sign and was inapplicable in the original problem, is here the true result.

Exercise XC.

1. The sum of the squares of three consecutive numbers is 365. Find the numbers.

2. Three times the product of two consecutive numbers exceeds four times their sum by 8. Find the numbers.

3. The product of three consecutive numbers is equal to three times the middle number. Find the numbers.

4. A boy bought a number of apples for 16 cents. Had he bought 4 more for the same money he would have paid $\frac{1}{3}$ of a cent less for each apple. How many did he buy?

5. For building 108 rods of stone-wall, 6 days less would have been required if 3 rods more a day had been built. How many rods a day were built?

6. A merchant bought some pieces of silk for \$900. Had he bought 3 pieces more for the same money he would have paid \$15 less for each piece. How many did he buy?

7. A merchant bought some pieces of cloth for \$168.75. He sold the cloth for \$12 a piece and gained as much as 1 piece cost him. How much did he pay for each piece?

8. Find the price of eggs per score when 10 more in $62\frac{1}{2}$ cents' worth lowers the price $31\frac{1}{4}$ cents per hundred.

9. The area of a square may be doubled by increasing its length by 6 inches and its breadth by 4 inches. Determine its side.

10. The length of a rectangular field exceeds the breadth by 1 yard, and the area is 3 acres. Determine its dimensions.

11. There are three lines of which two are each $\frac{4}{7}$ of the third, and the sum of the squares described on them is equal to a square yard. Determine the lengths of the lines in inches.

12. A grass plot 9 yards long and 6 yards broad has a path round it. The area of the path is equal to that of the plot. Determine the width of the path.

13. Find the radius of a circle the area of which would be doubled by increasing its radius by 1 inch.

14. Divide a line 20 inches long into two parts so that the rectangle contained by the whole and one part may be equal to the square on the other part.

15. A can do some work in 9 hours less time than B can do it, and together they can do it in 20 hours. How long will it take each alone to do it?

16. A vessel which has two pipes can be filled in 2 hours less time by one than by the other, and by both together in 2 hours 55 minutes. How long will it take each pipe alone to fill the vessel?

17. A vessel which has two pipes can be filled in 2 hours less time by one than by the other, and by both together in 1 hour 52 minutes 30 seconds. How long will it take each pipe alone to fill the vessel?

18. An iron bar weighs 36 pounds. If it had been 1 foot longer each foot would have weighed $\frac{1}{2}$ a pound less. Find the length and the weight per foot.

19. A number is expressed by two digits, the second of which is the square of the other, and when 54 is added its digits are interchanged. Find the number.

20. Divide 35 into two parts so that the sum of the two fractions formed by dividing each part by the other may be $2\frac{1}{12}$.

21. A boat's crew row $3\frac{1}{2}$ miles down a river and back again in 1 hour 40 minutes. If the current of the river is 2 miles per hour, determine their rate of rowing in still water.

22. A detachment from an army was marching in regular column with 5 men more in depth than in front. On approaching the enemy the front was increased by 845 men, and the whole was thus drawn up in 5 lines. Find the number of men.

23. A jockey sold a horse for \$144, and gained as much per cent as the horse cost. What did the horse cost?

24. A merchant expended a certain sum of money in goods, which he sold again for \$24, and lost as much per cent as the goods cost him. How much did he pay for the goods?

25. A broker bought a number of bank shares (\$100 each), when they were at a certain per cent *discount*, for \$7500; and afterwards when they were at the same per cent *premium*, sold all but 60 for \$5000. How many shares did he buy, and at what price?

26. The thickness of a rectangular solid is $\frac{2}{3}$ of its width, and its length is equal to the sum of its width and thickness; also, the number of cubic yards in its volume added to the number of linear yards in its edges is $\frac{5}{3}$ of the number of square yards in its surface. Determine its dimensions.

27. If a carriage-wheel $16\frac{1}{2}$ feet round took 1 second more to revolve, the rate of the carriage per hour would be $1\frac{7}{8}$ miles less. At what rate is the carriage travelling?

CHAPTER XV.

SIMULTANEOUS QUADRATIC EQUATIONS.

237. Quadratic equations involving *two* unknown quantities require different methods for their solution, according to the *form* of the equations.

238. CASE I. When from one of the equations the value of one of the unknown quantities can be found in terms of the other, and this value *substituted* in the other equation.

Ex. Solve: $$\left.\begin{aligned} 3x^2 - 2xy &= 5 \\ x - y &= 2 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Transpose x in (2),	$y = x - 2.$
Substitute in (1),	$3x^2 - 2x(x - 2) = 5.$
The solution of which gives	$x = 1$ or $-5.$
	$\therefore y = -1$ or $-7.$

Special methods often give more elegant solutions of examples than the *general* method by *substitution*.

I. *When equations have the form,* $x \pm y = a$, *and* $xy = b$; $x^2 \pm y^2 = a$; *and* $xy = b$; *or,* $x \pm y = a$, *and* $x^2 + y^2 = b$.

(1) Solve: $$\left.\begin{aligned} x + y &= 40 \\ xy &= 300 \end{aligned}\right\} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

Square (1),	$x^2 + 2xy + y^2 = 1600.$	(3)
Multiply (2) by 4,	$4xy \quad = 1200.$	(4)
Subtract (4) from (3),	$x^2 - 2xy + y^2 = 400.$	
Extract root of each side.	$x - y = \pm 20.$	(6)
Add (1) and (6),	$2x = 60$ or $20,$	
	$\therefore x = 30$ or $10.$	
Subtract (6) from (1),	$2y = 20$ or $60,$	
	$\therefore y = 10$ or $30.$	

(2) Solve:
$$\left.\begin{aligned} x - y &= 4 \\ x^2 + y^2 &= 40 \end{aligned}\right\}\qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Square (1), $x^2 - 2xy + y^2 = 16.$ (3)
Subtract (2) from (3), $-2xy = -24.$ (4)
Subtract (4) from (2), $x^2 + 2xy + y^2 = 64.$
Extract the root, $x + y = \pm 8.$ (5)
By combining (5) and (1), $x = 6$ or $-2.$
$y = 2$ or $-6.$

(3) Solve:
$$\left.\begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{9}{20} \\ \frac{1}{x^2} + \frac{1}{y^2} &= \frac{41}{400} \end{aligned}\right\}\qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Square (1), $\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = \frac{81}{400}.$ (3)

Subtract (2) from (3), $\frac{2}{xy} = \frac{40}{400}.$ (4)

Subtract (4) from (2), $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = \frac{1}{400}.$

Extract the root, $\frac{1}{x} - \frac{1}{y} = \pm\frac{1}{20}.$ (5)

By combining (1) and (5), $x = 4$ or $5.$
$y = 5$ or $4.$

II. *When one equation may be simplified by dividing it by the other.*

(4) Solve:
$$\left.\begin{aligned} x^3 + y^3 &= 91 \\ x + y &= 7 \end{aligned}\right\}\qquad \begin{matrix}(1)\\(2)\end{matrix}$$

Divide (1) by (2), $x^2 - xy + y^2 = 13.$ (3)
Square (2), $x^2 + 2xy + y^2 = 49.$ (4)
Subtract (3) from (4), $3xy = 36.$
Divide by -3, $-xy = -12.$ (5)
Add (5) and (3), $x^2 - 2xy + y^2 = 1.$
Extract the root, $x - y = \pm 1.$ (6)
By combining (6) and (2), $x = 4$ or $3.$
$y = 3$ or $4.$

Exercise XCI.

Solve:

1. $\left.\begin{array}{l} x+y=13 \\ xy=36 \end{array}\right\}$

2. $\left.\begin{array}{l} x+y=29 \\ xy=100 \end{array}\right\}$

3. $\left.\begin{array}{l} x-y=19 \\ xy=66 \end{array}\right\}$

4. $\left.\begin{array}{l} x-y=45 \\ xy=250 \end{array}\right\}$

5. $\left.\begin{array}{l} x-y=10 \\ x^2+y^2=178 \end{array}\right\}$

6. $\left.\begin{array}{l} x-y=14 \\ x^2+y^2=436 \end{array}\right\}$

7. $\left.\begin{array}{l} x+y=12 \\ x^2+y^2=104 \end{array}\right\}$

8. $\left.\begin{array}{l} \frac{1}{x}+\frac{1}{y}=\frac{3}{4} \\ \frac{1}{x^2}+\frac{1}{y^2}=\frac{5}{16} \end{array}\right\}$

9. $\left.\begin{array}{l} \frac{1}{x}+\frac{1}{y}=5 \\ \frac{1}{x^2}+\frac{1}{y^2}=13 \end{array}\right\}$

10. $\left.\begin{array}{l} 7x^2-8xy=159 \\ 5x+2y=7 \end{array}\right\}$

11. $\left.\begin{array}{l} x+y=49 \\ x^2+y^2=1681 \end{array}\right\}$

12. $\left.\begin{array}{l} x^3+y^3=341 \\ x+y=11 \end{array}\right\}$

13. $\left.\begin{array}{l} x^3+y^3=1008 \\ x+y=12 \end{array}\right\}$

14. $\left.\begin{array}{l} x^3-y^3=98 \\ x-y=2 \end{array}\right\}$

15. $\left.\begin{array}{l} x^3-y^3=279 \\ x-y=3 \end{array}\right\}$

16. $\left.\begin{array}{l} x-3y=1 \\ xy+y^2=5 \end{array}\right\}$

17. $\left.\begin{array}{l} 4y=5x+1 \\ 2xy=33-x^2 \end{array}\right\}$

18. $\left.\begin{array}{l} \frac{1}{x}-\frac{1}{y}=3 \\ \frac{1}{x^2}-\frac{1}{y^2}=21 \end{array}\right\}$

19. $\left.\begin{array}{l} \frac{1}{x}-\frac{1}{y}=2\frac{1}{2}. \\ \frac{1}{x^2}-\frac{1}{y^2}=8\frac{3}{4} \end{array}\right\}$

20. $\left.\begin{array}{l} x^2-2xy-y^2=1 \\ x+y=2 \end{array}\right\}$

239. CASE II. When each of the two equations is *homogeneous* and of the *second degree*.

Ex. Solve:
$$\left.\begin{array}{l} 2y^2-4xy+3x^2=17 \\ y^2-x^2=16 \end{array}\right\} \begin{array}{r}(1)\\(2)\end{array}$$

Let $y = vx$, and substitute vx for y in both equations.

From (1), $2v^2x^2 - 4vx^2 + 3x^2 = 17$,

$$\therefore x^2 = \frac{17}{2v^2 - 4v + 3}.$$

From (2), $v^2x^2 - x^2 = 16$,

$$\therefore x^2 = \frac{16}{v^2 - 1}.$$

Equate the values of x^2,

$$\frac{17}{2v^2-4v+3} = \frac{16}{v^2-1},$$

$$32v^2 - 64v + 48 = 17v^2 - 17,$$

$$15v^2 - 64v = -65.$$

The solution gives, $v = \frac{5}{3}$ or $\frac{13}{5}$.

Substitute the value of v in $x^2 = \frac{16}{v^2-1}$,

then, $x^2 = 9$ or $\frac{25}{9}$,

$$\therefore x = \pm 3 \text{ or } \pm\frac{5}{3},$$

and $y = vx = \pm 5$ or $\pm\frac{13}{3}$.

EXERCISE XCII.

Solve:

1. $\left.\begin{array}{l} x^2+xy+2y^2=74 \\ 2x^2+2xy+y^2=73 \end{array}\right\}$

2. $\left.\begin{array}{l} x^2+xy+4y^2=6 \\ 3x^2+8y^2=14 \end{array}\right\}$

3. $\left.\begin{array}{l} x^2-xy+y^2=21 \\ y^2-2xy=-15 \end{array}\right\}$

4. $\left.\begin{array}{l} x^2-4y^2-9=0 \\ xy+2y^2-3=0 \end{array}\right\}$

5. $\left.\begin{array}{l} x^2-xy-35=0 \\ xy+y^2-18=0 \end{array}\right\}$

6. $\left.\begin{array}{l} x^2+xy+2y^2=44 \\ 2x^2-xy+y^2=16 \end{array}\right\}$

7. $\left.\begin{array}{l} x^2+xy-15=0 \\ xy-y^2-2=0 \end{array}\right\}$

8. $\left.\begin{array}{l} x^2-xy+y^2=7 \\ 3x^2+13xy+8y^2=162 \end{array}\right\}$

9. $\left.\begin{array}{l} 2x^2+3xy+y^2=70 \\ 6x^2+xy-y^2=50 \end{array}\right\}$

10. $\left.\begin{array}{l} x^2-xy-y^2=5 \\ 2x^2+3xy+y^2=28 \end{array}\right\}$

240. Case III. When the two equations are *symmetrical* with respect to x and y; that is, when they have x and y similarly involved in them.

Thus, the expressions $2x^3+3x^2y^2+2y^3$, $2xy-3x-3y+1$, $x^4-3x^3y-3xy^3+y^4$ are symmetrical expressions.

(1) Solve: $\left.\begin{array}{l} x^3+y^3=18xy \\ x+y=12 \end{array}\right\}$ $\quad$ (1) (2)

Put $u+v$ for x, and $u-v$ for y, in (1) and (2).

(1) becomes $(u+v)^3+(u-v)^3=18(u+v)(u-v)$,

or $u^3+3uv^2=9(u^2-v^2)$. (3)

(2) becomes $(u+v)+(u-v)=12$,

or $2u=12$,

$\therefore u=6$.

Substitute 6 for u in (3).

(3) becomes $216+18v^2=9(36-v^2)$,

whence, $v^2=4$,

$\therefore v=\pm 2$,

$\therefore x=u+v=6\pm 2=8$ or 4,

and $y=u-v=6\mp 2=4$ or 8.

(2) Solve: $\left.\begin{array}{l} x+y=8 \\ x^4+y^4=706 \end{array}\right\}$ $\quad$ (1) (2)

Put $u+v$ for x, and $u-v$ for y, in (1) and (2).

(1) becomes $(u+v)+(u-v)=8$,

$\therefore u=4$.

(2) becomes $u^4+6u^2v^2+v^4=353$. (3)

Substitute 4 for u in (3),

$256+96v^2+v^4=353$,

or, $v^4+96v^2=97$. (4)

The solution of (4) gives $v=\pm 1$ or $\pm\sqrt{-97}$.

Taking the possible values of v, $x=5$ or 3, and $y=3$ or 5.

Exercise XCIII.

Solve:

1. $\left.\begin{aligned} 4xy &= 96 - x^2y^2 \\ x + y &= 6 \end{aligned}\right\}$

2. $\left.\begin{aligned} x^2 + y^2 &= 18 - x - y \\ xy &= 6 \end{aligned}\right\}$

3. $\left.\begin{aligned} 2(x^2 + y^2) &= 5xy \\ 4(x - y) &= xy \end{aligned}\right\}$

4. $\left.\begin{aligned} 4(x + y) &= 3xy \\ x + y + x^2 + y^2 &= 26 \end{aligned}\right\}$

5. $\left.\begin{aligned} 4x^2 + xy + 4y^2 &= 58 \\ 5x^2 + 5y^2 &= 65 \end{aligned}\right\}$

6. $\left.\begin{aligned} xy(x + y) &= 30 \\ x^3 + y^3 &= 35 \end{aligned}\right\}$

241. The preceding cases are *general methods* for the solution of equations which belong to the kinds referred to; often, however, in the solution of these and other kinds of simultaneous equations involving quadratics, a little ingenuity will suggest some step by which the roots may easily be found.

Exercise XCIV.

Solve:

1. $\left.\begin{aligned} x - y &= 7 \\ x^2 + xy + y^2 &= 13 \end{aligned}\right\}$

2. $\left.\begin{aligned} x^2 + xy &= 35 \\ xy - y^2 &= 6 \end{aligned}\right\}$

3. $\left.\begin{aligned} xy - 12 &= 0 \\ x - 2y &= 5 \end{aligned}\right\}$

4. $\left.\begin{aligned} xy - 7 &= 0 \\ x^2 + y^2 &= 50 \end{aligned}\right\}$

5. $\left.\begin{aligned} 2x - 5y &= 9 \\ x^2 - xy + y^2 &= 7 \end{aligned}\right\}$

6. $\left.\begin{aligned} x - y &= 9 \\ xy + 8 &= 0 \end{aligned}\right\}$

7. $\left.\begin{aligned} 5x - 7y &= 0 \\ 5x^2 - \frac{13xy}{4} &= 4 - 7y^2 \end{aligned}\right\}$

8. $\left.\begin{aligned} x - y &= 1 \\ x^2 + y^2 &= 8\tfrac{1}{2} \end{aligned}\right\}$

9. $\left.\begin{aligned} x^2 + 4xy &= 3 \\ 4xy + y^2 &= 2\tfrac{1}{4} \end{aligned}\right\}$

10. $\left.\begin{aligned} x^2 - xy + y^2 &= 48 \\ x - y - 8 &= 0 \end{aligned}\right\}$

11. $\left.\begin{aligned} x^2 + 3xy + y^2 &= 1 \\ 3x^2 + xy + 3y^2 &= 13 \end{aligned}\right\}$

12. $\left.\begin{aligned} x^2 - 2xy + 3y^2 &= 1\tfrac{2}{9} \\ x^2 + xy - y^2 &= \tfrac{1}{9}. \end{aligned}\right\}$

13. $\left.\begin{aligned} x + y &= a \\ 4xy - a^2 &= -4b^2 \end{aligned}\right\}$

14. $\left.\begin{aligned} x - y &= 1 \\ \frac{x}{y} + \frac{y}{x} &= 2\tfrac{1}{6} \end{aligned}\right\}$

15. $\left.\begin{array}{l} x^2 + 9xy = 340 \\ 7xy - y^2 = 171 \end{array}\right\}$

16. $\left.\begin{array}{l} x + y = 6 \\ x^3 + y^3 = 72 \end{array}\right\}$

17. $\left.\begin{array}{l} 3xy + 2x + y = 485 \\ 3x - 2y = 0 \end{array}\right\}$

18. $\left.\begin{array}{l} x - y = 1 \\ x^3 - y^3 = 19 \end{array}\right\}$

19. $\left.\begin{array}{l} x^3 + y^3 = 2728 \\ x^2 - xy + y^2 = 124 \end{array}\right\}$

20. $\left.\begin{array}{l} x + y = a \\ x^2 + y^2 = b^2 \end{array}\right\}$

21. $\left.\begin{array}{l} x^2 - y^2 = 0 \\ 3x^2 - 4xy + 5y^2 = 9 \end{array}\right\}$

22. $\left.\begin{array}{l} \dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{10}{3} \\ x^2 + y^2 = 45 \end{array}\right\}$

23. $\left.\begin{array}{l} \dfrac{1}{x} + \dfrac{1}{y} = 5 \\ \dfrac{1}{x+1} + \dfrac{1}{y+1} = \dfrac{17}{12} \end{array}\right\}$

24. $\left.\begin{array}{l} x^2 - xy + y^2 = 7 \\ x^4 + x^2y^2 + y^4 = 133 \end{array}\right\}$

25. $\left.\begin{array}{l} x + y = 4 \\ x^4 + y^4 = 82 \end{array}\right\}$

26. $\left.\begin{array}{l} x^3 - y^3 = a^3 \\ x - y = a \end{array}\right\}$

27. $\left.\begin{array}{l} x^2 - xy = a^2 + b^2 \\ xy - y^2 = 2ab \end{array}\right\}$

28. $\left.\begin{array}{l} x^2 - y^2 = 4ab \\ xy = a^2 - b^2 \end{array}\right\}$

29. $\left.\begin{array}{l} xy = 0 \\ x^2 + y^2 = 16 \end{array}\right\}$

30. $\left.\begin{array}{l} x^2 + xy + y^2 = 37 \\ x^4 + x^2y^2 + y^4 = 481 \end{array}\right\}$

31. $\left.\begin{array}{l} x^2 = ax + by \\ y^2 = ay + bx \end{array}\right\}$

32. $\left.\begin{array}{l} x - y - 2 = 0 \\ 15(x^2 - y^2) = 16xy \end{array}\right\}$

33. $\left.\begin{array}{l} \dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{89}{40} \\ 6x = 20y + 9 \end{array}\right\}$

34. $\left.\begin{array}{l} \dfrac{x}{a} + \dfrac{y}{b} = 1 \\ \dfrac{a}{x} + \dfrac{b}{y} = 4 \end{array}\right\}$

35. $\left.\begin{array}{l} x^2 + y^2 = 7 + xy \\ x^3 + y^3 = 6xy - 1 \end{array}\right\}$

36. $\left.\begin{array}{l} x^5 - y^5 = 3093 \\ x - y = 3 \end{array}\right\}$

37. $\left.\begin{array}{l} \frac{3}{5}(x-1) - \frac{2}{3}(x+1)(y-1) = -11 \\ \frac{1}{3}(y+2) = \frac{1}{4}(x+2) \end{array}\right\}$

38. $\left.\begin{array}{l} 10x^2 + 15xy = 3ab - 2a^2 \\ 10y^2 + 15xy = 3ab - 2b^2 \end{array}\right\}$

Exercise XCV.

1. If the length and breadth of a rectangle were each increased by 1, the area would be 48; if they were each diminished by 1, the area would be 24. Find the length and breadth.

2. The sum of the squares of the two digits of a number is 25, and the product of the digits is 12. Find the number.

3. The sum, the product, and the difference of the squares, of two numbers are all equal. Find the numbers.

   NOTE. Represent the numbers by $x+y$ and $x-y$, respectively.

4. The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356. What are the numbers?

5. The numerator and denominator of one fraction are each greater by 1 than those of another, and the sum of the two fractions is $1\frac{5}{12}$; if the numerators were interchanged the sum of the fractions would be $1\frac{1}{2}$. Find the fractions.

6. A man starts from the foot of a mountain to walk to its summit. His rate of walking during the second half of the distance is $\frac{1}{2}$ mile per hour less than his rate during the first half, and he reaches the summit in $5\frac{1}{2}$ hours. He descends in $3\frac{3}{4}$ hours, by walking 1 mile more per hour than during the first half of the ascent. Find the distance to the top and the rates of walking.

NOTE. Let $2x =$ the distance, and y miles per hour $=$ the rate at first.

Then $\frac{x}{y} + \frac{x}{y-\frac{1}{2}} = 5\frac{1}{2}$ hours, and $\frac{2x}{y+1} = 3\frac{3}{4}$ hours.

7. The sum of two numbers which are formed by the same two digits in reverse order is $\frac{55}{18}$ of their difference; and the difference of the squares of the numbers is 3960. Determine the numbers.

8. The hypotenuse of a right triangle is 20, and the area of the triangle is 96. Determine the sides.

The equation $x = 6 + 4m$ shows that m in respect ... but cannot have a negative value greater than 1.

$$\therefore m \text{ may be } 0 \text{ or } -1,$$

and then $\quad x = 6,\ y = 1,$

or $\quad x = 2,\ y = 4.$

(2) Solve $5x - 14y = 11$, in positive integers.

Transpose, $\quad 5x = 11 + 14y,$

$$x = 2 + 2y + \frac{1 + 4y}{5},$$

$$\therefore x - 2y - 2 = \frac{1 + 4y}{5},$$

$$\therefore \frac{1 + 4y}{5} \text{ must be integral.}$$

Now, if $\frac{1 + 4y}{5}$ be put $= m$, then $y = \frac{5m - 1}{4}$, a fraction in form.

post?

11. The fore wheel of a carriage turns in a mile 132 times more than the hind wheel; but if the circumferences were each increased by 2 feet, it would turn only 88 times more. Find the circumference of each.

12. A person has $6500, which he divides into two parts and loans at *different rates* of interest, so that the two parts produce *equal* returns. If the first part had been loaned at the second rate of interest, it would have produced $180; and if the second part had been loaned at the first rate of interest, it would have produced $245. Find the rates of interest.

Exercise XCV.

1. If the length and breadth of a rectangle were each increased by 1, the area would be 48; if they were each diminished by 1, the area would be 24. Find the length and breadth.

2. The sum of the squares of the two digits of a number is 25, and the product of the digits is 12. Find the number.

3. The sum, the product, and the difference of the squares, of two numbers are all equal. Find the numbers.

   Note. Represent the numbers by $x+y$ and $x-y$, respectively.

4. The difference of two numbers is $\frac{3}{8}$ of the greater, and the sum of their squares is 356. What are the numbers?

5. The numerator and denominator of one fraction are each greater by 1 than those of another, and the sum of the two fractions is $1\frac{5}{12}$; if the numerators were in-terchanged the

244. The method of solving an indeterminate equation in positive integers is as follows:

(1) Solve $3x+4y=22$, in positive integers.

Transpose, $\quad 3x = 22 - 4y,$

$$\therefore x = 7 - y + \frac{1-y}{3},$$

the quotient being written as a mixed expression.

$$\therefore x + y - 7 = \frac{1-y}{3}.$$

Since the values of x and y are to be integral, $x+y-7$ will be integral, and hence, $\frac{1-y}{3}$ will be integral, though written in the *form* of a fraction.

Let $\quad \frac{1-y}{3} = m$, an integer;

7. The sum of two numbers which are formed by the same two digits in reverse order is $\frac{5}{11}$ of their difference, and the difference of the squares of the numbers is 3960. Determine the numbers.

8. The hypotenuse of a right triangle is 20, and the area [illegible] Determine the sides.

n.

al equation,

m.

ı respect to y may be 0,
a positive value.

The equation $x = 6 + 4m$ shows that m in respect to x may be 0, but cannot have a negative value greater than 1.

$$\therefore m \text{ may be } 0 \text{ or } -1,$$

and then $\qquad x = 6,\ y = 1,$

or $\qquad x = 2,\ y = 4.$

(2) Solve $5x - 14y = 11$, in positive integers.

Transpose, $\qquad 5x = 11 + 14y,$

$$x = 2 + 2y + \frac{1 + 4y}{5},$$

$$\therefore x - 2y - 2 = \frac{1 + 4y}{5},$$

$$\therefore \frac{1 + 4y}{5} \text{ must be integral.}$$

Now, if $\frac{1 + 4y}{5}$ be put $= m$, then $y = \frac{5m - 1}{4}$, a fraction in form.

To avoid this difficulty, it is necessary in some way to make the coefficient of y equal to *unity*. Since $\frac{1 + 4y}{5}$ is integral, any multiple of $\frac{1 + 4y}{5}$ is integral. Multiply, then, by such a number as will make the coefficient of y *greater by* 1 than some multiple of the denominator. In this case, multiply by 4. Then

$$\frac{4 + 16y}{5} \text{ or } 3y + \frac{4 + y}{5} \text{ is integral}$$

$$\therefore \frac{4 + y}{5} = m, \text{ an integer};$$

$$\therefore y = 5m - 4.$$

Since $\qquad x = \frac{1}{5}(11 + 14y)$, from the original equation,

$$\therefore x = 14m - 9.$$

Here it is obvious that m may have *any positive value*, and

$$x = 5,\ 19,\ 33.....$$
$$y = 1,\ 6,\ 11.....$$

The required multiplier can always be found when the coefficients are prime to each other, and it is best to divide the original equation by the smaller of the two coefficients, in order to have the multiplier as small as possible.

245. The necessity for a multiplier may often be obviated by a little ingenuity. Thus,

The equation $4y = 29 - 7x$ may be put in the form of

$$4y = 29 - 8x + x,$$
$$\therefore y = 7 - 2x + \frac{1+x}{4},$$

in which the fraction is of the required form.

The equation $5x = 18 + 13y$

gives $$x = 3 + 2y + \frac{3(1+y)}{5},$$

in which $\frac{1+y}{5}$ is of the required form.

246. It will be seen from (1) and (2) that when only *positive integers* are required, the number of solutions will be *limited* or *unlimited* according as the sign connecting x and y is *positive* or *negative.*

(3) Find the least number that when divided by 14 and 5 will give remainders 1 and 3 respectively.

If N represent the number, then

$$\frac{N-1}{14} = x, \text{ and } \frac{N-3}{5} = y,$$
$$\therefore N = 14x + 1, \text{ and } N = 5y + 3,$$
$$\therefore 14x + 1 = 5y + 3.$$
$$5y = 14x - 2,$$
$$5y = 15x - 2 - x,$$
$$\therefore y = 3x - \frac{2+x}{5}.$$

Let $\frac{2+x}{5} = m$, an integer;

$$\therefore x = 5m - 2.$$

$y = \frac{1}{5}(14x - 2)$, from original equation,

$$\therefore y = 14m - 6.$$

If $m = 1$, $x = 3$, and $y = 8$,

$$\therefore N = 14x + 1 = 5y + 3 = 43. \ \textit{Ans.}$$

(4) Solve $5x + 6y = 30$, so that x may be a multiple of y, and both positive.

Let $\qquad x = my.$

Then $\qquad (5m + 6)y = 30,$

$$\therefore y = \frac{30}{5m + 6},$$

and $\qquad x = \dfrac{30m}{5m + 6}.$

If $m = 2$, $\qquad x = 3\frac{3}{4},\ y = 1\frac{7}{8}.$

If $m = 3$, $\qquad x = 4\frac{2}{7},\ y = 1\frac{3}{7}.$

(5) Solve $14x + 22y = 71$, in positive integers.

$$x = 5 - y + \frac{1 - 8y}{14}.$$

If we multiply the fraction by 7 and reduce,

the result is $\qquad -4y + \frac{1}{2},$

a form which shows that there can be no *integral* solution.

There can be no integral solution of $ax \pm by = c$ if a and b have a common factor not common also to c; for, if d be a factor of a and also of b, but not of c, the equation may be written,

$$mdx \pm ndy = c,$$

or $\qquad mx \pm ny = \dfrac{c}{d}$, a fraction.

Exercise XCVI.

Solve in positive integers:

1. $2x + 11y = 49.$
2. $7x + 3y = 40.$
3. $5x + 7y = 53.$
4. $x + 10y = 29.$
5. $3x + 8y = 61.$
6. $8x + 5y = 97.$
7. $16x + 7y = 110.$
8. $7x + 10y = 206.$

Solve in least positive integers:

9. $12x - 7y = 1.$
10. $5x - 17y = 23.$
11. $23y - 13x = 3.$
12. $23x - 9y = 929.$
13. $23x - 33y = 43.$
14. $555x - 22y = 73.$

15. How many fractions are there with denominators 12 and 18 whose sum is $\frac{25}{36}$?

16. What is the least number which, when divided by 3 and 5, leaves remainders 2 and 3 respectively?

17. A person counting a basket of eggs, which he knows are between 50 and 60, finds that when he counts them 3 at a time there are 2 over; but when he counts them 5 at a time there are 4 over. How many are there in all?

18. A person bought 40 animals, consisting of pigs, geese, and chickens, for \$40. The pigs cost \$5 a piece, the geese \$1, and the chickens 25 cents each. Find the number he bought of each.

19. Find the least multiple of 7 which, when divided by 2, 3, 4, 5, 6, leaves in each case 1 for a remainder.

20. In how many ways may 100 be divided into two parts, one of which shall be a multiple of 7 and the other of 9?

21. Solve $18x - 5y = 70$ so that y may be a multiple of x, and both positive.

22. Solve $8x + 12y = 23$ so that x and y may be positive, and their sum an integer.

23. Divide 70 into three parts which shall give integral quotients when divided by 6, 7, 8, respectively, and the sum of the quotients shall be 10.

24. Divide 200 into three parts which shall give integral quotients when divided by 5, 7, 11, respectively, and the sum of the quotients shall be 20.

25. A number consisting of three digits, of which the middle one is 4, has the digits in the units' and hundreds' places interchanged by adding 792. Find the number.

26. Some men earning each \$2.50 a day, and some women earning each \$1.75 a day, receive altogether for their daily wages \$44.75. Determine the number of men and the number of women.

27. A wishes to pay B a debt of £1 12*s*., but has only half-crowns in his pocket, while B has only 4 penny-pieces. How may they settle the matter most simply?

28. Show that $323x - 527y = 1000$ cannot be satisfied by integral values of x and y.

29. A farmer buys oxen, sheep, and hens. The whole number bought is 100, and the whole price £100. If the oxen cost £5, the sheep £1, and the hens 1*s*. each, how many of each did he buy?

30. A number of lengths 3 feet, 5 feet, and 8 feet are cut; how may 48 of them be taken so as to measure 175 feet all together?

31. A field containing an integral number of acres less than 10 is divided into 8 lots of one size, and 7 of 4 times that size, and has also a road passing through it containing 1300 square yards. Find the size of the lots in square yards.

32. Two wheels are to be made, the circumference of one of which is to be a multiple of the other. What circumferences may be taken so that when the first has gone round three times and the other five, the difference in the length of rope coiled on them may be 17 feet?

33. In how many ways can a person pay a sum of £15 in half-crowns, shillings, and sixpences, so that the number of shillings and sixpences together shall be equal to the number of half-crowns?

CHAPTER XVII.

INEQUALITIES.

247. Expressions containing any given letter will have their values changed when different values are assigned to that letter; and of two such expressions, one may be for some values of the letter larger than the other, for other values of the letter smaller than the other.

Thus, $1 + x + x^2$ will be greater than $1 - x + x^2$ for all positive values of x, but less for all negative values of x.

248. One expression, however, may be so related to another that, whatever values may be given to the letter, it cannot be greater than the other.

Thus, $2x$ cannot be greater than $x^2 + 1$, whatever value be given to x.

249. For finding whether this relation holds between two expressions, the following is a fundamental proposition:

If a and b are unequal, $a^2 + b^2 > 2ab$.

For, $(a - b)^2$ must be positive, whatever the values of a and b.

That is, $$(a - b)^2 > 0,$$

or $$a^2 - 2ab + b^2 > 0;$$

$$\therefore a^2 + b^2 > 2ab.$$

250. The principles applied to the solution of equations may be applied to inequalities, except that if each side of an equality have its *sign changed*, the inequality will be *reversed*.

Thus, if $a > b$, then $-a$ will be $< -b$.

(1) If a and b be positive, show that a^3+b^3 is $> a^2b+ab^2$.

$$a^3+b^3 > a^2b+ab^2,$$

if (dividing each side by $a+b$),

$$a^2-ab+b^2 > ab,$$

if $$a^2+b^2 > 2ab.$$

But $$a^2+b^2 \text{ is } > 2ab, \qquad \text{§ 249.}$$

$$\therefore a^3+b^3 > a^2b+ab^2.$$

(2) Show that $a^2+b^2+c^2$ is $> ab+ac+bc$.

Now, $$a^2+b^2 \text{ is } > 2ab,$$
$$a^2+c^2 \text{ is } > 2ac, \qquad \text{§ 249.}$$
$$b^2+c^2 \text{ is } > 2bc.$$

By adding, $2a^2+2b^2+2c^2$ is $> 2ab+2ac+2bc$,

$$\therefore a^2+b^2+c^2 \text{ is } > ab+ac+bc.$$

Exercise XCVII.

Show that, the letters being unequal and positive:

1. a^2+3b^2 is $> 2b(a+b)$.   2. a^3b+ab^3 is $> 2a^2b^2$.
3. $(a^2+b^2)(a^4+b^4)$ is $> (a^3+b^3)^2$.
4. $a^2b+a^2c+ab^2+b^2c+ac^2+bc^2$ is $> 6abc$.
5. The sum of any fraction and its reciprocal is > 2.
6. If $x^2=a^2+b^2$, and $y^2=c^2+d^2$, xy is $> ac+bd$, or $ad+bc$.
7. $ab+ac+bc < (a+b-c)^2+(a+c-b)^2+(b+c-a)^2$.
8. Which is the greater, $(a^2+b^2)(c^2+d^2)$ or $(ac+bd)^2$?
9. Which is the greater, m^2+m or m^3+1?
10. Which is the greater, a^4-b^4 or $4a^3(a-b)$ when a is $> b$?
11. Which is the greater, $\sqrt{\dfrac{a^2}{b}}+\sqrt{\dfrac{b^2}{a}}$ or $\sqrt{a}+\sqrt{b}$?
12. Which is the greater, $\dfrac{a+b}{2}$ or $\dfrac{2ab}{a+b}$?
13. Which is the greater, $\dfrac{a}{b^2}+\dfrac{b}{a^2}$ or $\dfrac{1}{b}+\dfrac{1}{a}$?

CHAPTER XVIII.

THEORY OF EXPONENTS.

251. The expression a^n, when n is a positive integer, has been defined as the product of n equal factors each equal to a. § 24.

And it has been shown that $a^m \times a^n = a^{m+n}$. § 66.

That $a^m \div a^n = a^{m-n}$, if m be greater than n; § 93.

or $\frac{1}{a^{n-m}}$, if m be less than n. § 94.

And that $(a^m)^n = a^{mn}$. § 199.

Also, it is true that $a^n \times b^n = (ab)^n$; for

$(ab)^n = ab$ taken n times as a factor,

$= a$ taken n times as a factor $\times$ b taken n times as a factor $= a^n \times b^n$.

252. Likewise, $\sqrt[n]{a}$, when n is a positive integer, has been defined as *one* of the n *equal factors* of a (§ 203); so that if $\sqrt[n]{a}$ be taken n times as a factor, the resulting product is a; that is, $(\sqrt[n]{a})^n = a$.

Again, the expression $\sqrt[n]{a^m}$ means that a is to be raised to the *mth* power, and the *nth* root of the result obtained.

And the expression $(\sqrt[n]{a})^m$ means that the *nth* root of a is to be taken, and the result raised to the *mth* power.

It will thus be seen that any proposition relating to roots and powers may be expressed by this method of notation. It is, however, *found convenient* to adopt another method of notation, in which fractional and negative exponents are used.

253. The meaning of a fractional exponent is at once suggested, by observing that the *division of an exponent*, when the resulting quotient is *integral*, is equivalent to extracting a root. Thus, a^3 is the square root of a^6, and 3, the exponent of a^3, is obtained by dividing the exponent of a^6 by 2.

If this division be indicated only, the square root of a^6 will be denoted by $a^{\frac{6}{2}}$, in which the *denominator* denotes the *root*, and the *numerator* the *power*. If the same meaning be given to an exponent when the division does not give an integral quotient, $a^{\frac{3}{2}}$ will represent the square root of the cube of a; and, in general, $a^{\frac{m}{n}}$, the *nth* root of the *mth* power of a. This, then, is the meaning that will be assigned to a fractional exponent, so that in a fractional exponent

254. *The numerator will indicate a power, and the denominator a root.*

255. The meaning of a negative exponent is suggested by observing that in a series of descending powers of a,

$$a^n \ldots\ldots a^5,\ a^4,\ a^3,\ a^2,\ a^1,$$

the subtraction of 1 from the exponent is equivalent to dividing by a; and if the operation be continued, the result is

$$a^0,\ a^{-1},\ a^{-2},\ a^{-3},\ a^{-4} \ldots\ldots a^{-n}.$$

Then
$$a^0 = \frac{a}{a} = 1;\quad a^{-1} = 1 \div a = \frac{1}{a};$$

$$a^{-2} = \frac{1}{a} \div a = \frac{1}{a^2};\quad a^{-n} = \frac{1}{a^n}.$$

This, then, is the meaning that will be assigned to a negative exponent, so that,

256. *A number with a negative exponent will denote the reciprocal of the number with the corresponding positive exponent.*

It may be easily shown that the laws which apply to positive integral exponents apply also to fractional and negative exponents.

257. To show that $a^{\frac{m}{n}} \times b^{\frac{m}{n}} = (ab)^{\frac{m}{n}}$:

$$\begin{aligned} a^{\frac{m}{n}} \times b^{\frac{m}{n}} &= \sqrt[n]{a^m} \times \sqrt[n]{b^m}, \\ &= \sqrt[n]{a^m b^m}, \\ &= \sqrt[n]{(ab)^m}, \\ &= (ab)^{\frac{m}{n}} \quad \text{(by definition).} \end{aligned}$$

Likewise $\quad a^{\frac{1}{n}} \times b^{\frac{1}{n}} \times c^{\frac{1}{n}} = (abc)^{\frac{1}{n}}$, and so on.

258. To show that $(a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}$:

Let $\quad x = (a^{\frac{1}{m}})^{\frac{1}{n}}.$

Then $\quad x^n = a^{\frac{1}{m}}$, and $x^{mn} = a$.

$\therefore x = a^{\frac{1}{mn}}.$

But $\quad x = (a^{\frac{1}{m}})^{\frac{1}{n}}$ (by supposition),

$\therefore (a^{\frac{1}{m}})^{\frac{1}{n}} = a^{\frac{1}{mn}}.$

259. To show that $a^m \times a^{-n} = a^{m-n}$:

Now $\quad a^m \times a^{-n} = a^m \times \frac{1}{a^n},$

$= \frac{a^m}{a^n} = a^{m-n}$ if $m > n$; § 93.

or $\quad = \frac{1}{a^{n-m}}$ if $m < n$, § 94.

$= a^{-(n-m)}$ (by definition),

$= a^{m-n}.$

260. In like manner the same laws may be shown to apply in every case.

5. $1 + ab^{-1} + a^2b^{-2}$ by $1 - ab^{-1} + a^2b^{-2}$.

6. $a^2b^{-2} + 2 + a^{-2}b^2$ by $a^2b^{-2} - 2 - a^{-2}b^2$.

7. $4x^{-3} + 3x^{-2} + 2x^{-1} + 1$ by $x^{-2} - x^{-1} + 1$.

Divide:

8. $x^{4n} - y^{4n}$ by $x^n - y^n$.

9. $x + y + z - 3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}}$.

10. $x + y$ by $x^{\frac{4}{5}} - x^{\frac{3}{5}}y^{\frac{1}{5}} + x^{\frac{2}{5}}y^{\frac{2}{5}} - x^{\frac{1}{5}}y^{\frac{3}{5}} + y^{\frac{4}{5}}$.

11. $x^2y^{-2} + 2 + x^{-2}y^2$ by $xy^{-1} + x^{-1}y$.

12. $a^{-4} + a^{-2}b^{-2} + b^{-4}$ by $a^{-2} - a^{-1}b^{-1} + b^{-2}$.

Find the squares of:

13. $4ab^{-1}$; $a^{\frac{1}{2}} - b^{\frac{1}{2}}$; $a + a^{-1}$; $2a^{\frac{1}{2}}b^{\frac{1}{3}} - a^{-\frac{1}{2}}b^{\frac{2}{3}}$.

If $a = 4$, $b = 2$, $c = 1$, find the values of:

14. $a^{\frac{1}{2}}b$; $5ab^{-1}$; $2(ab)^{\frac{1}{3}}$; $a^{-\frac{1}{2}}b^{-1}c^{\frac{2}{3}}$; $12a^{-2}b^{-3}$.

15. Expand $(a^{\frac{1}{2}} - b^{\frac{1}{3}})^3$; $(2x^{-1} + x)^4$; $(ab^{-1} - by^{-1})^6$.

Extract the square root of:

16. $9x^{-4} - 18x^{-3}y^{\frac{1}{2}} + 15x^{-2}y - 6x^{-1}y^{\frac{3}{2}} + y^2$.

Extract the cube root of:

17. $8x^3 + 12x^2 - 30x - 35 + 45x^{-1} + 27x^{-2} - 27x^{-3}$.

Resolve into prime factors with fractional exponents:

18. $\sqrt[3]{12}$, $\sqrt[4]{72}$, $\sqrt[6]{96}$, $\sqrt[8]{64}$; and find their product.

Simplify:

19. $\{(x^{5ab})^3 \times (x^{5b})^{-2}\}^{\frac{1}{3a-2}}$.   20. $(x^{18a} \times x^{-12})^{\frac{1}{3a-2}}$.

21. $3(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2 - 4(a^{\frac{1}{2}} + b^{\frac{1}{2}})(a^{\frac{1}{2}} - b^{\frac{1}{2}}) + (a^{\frac{1}{2}} - 2b^{\frac{1}{2}})^2$.

22. $\{(a^m)^{m - \frac{1}{m}}\}^{\frac{1}{m+1}}$.   24. $[\{(a^{-m})^{-n}\}^p]^q \div [\{(a^m)^n\}^{-p}]^{-q}$.

23. $\left(\dfrac{x^{p+q}}{x^q}\right)^p \div \left(\dfrac{x^q}{x^{q-p}}\right)^{p-q}$.   25. $\dfrac{x^{2p(q-1)} - y^{2q(p-1)}}{x^{p(q-1)} + y^{q(p-1)}}$.

Radical Expressions.

263. An indicated root that cannot be exactly obtained is called a **surd**, or **irrational number.** An indicated root that can be exactly obtained is said to have the *form* of a surd.

264. The required root shows the **order** of a surd; and surds are named *quadratic*, *cubic*, *biquadratic*, according as the *second*, *third*, or *fourth* roots are required.

265. The product of a rational factor and a surd factor is called a *mixed surd;* as, $3\sqrt{2}$, $b\sqrt{a}$.

266. When there is no rational factor outside of the radical sign, the surd is said to be *entire;* as, $\sqrt{2}$, $\sqrt{a}$.

267. Since $\sqrt[m]{a} \times \sqrt[m]{b} \times \sqrt[m]{c} = \sqrt[m]{abc}$, the product of two or more surds of the same order will be a radical expression of the same order consisting of the product of the numbers under the radical signs.

268. In like manner, $\sqrt{a^2b} = \sqrt{a^2} \times \sqrt{b} = a\sqrt{b}$. That is,

A factor under the radical sign whose root can be taken, may, by having the root taken, be removed from under the radical sign.

269. Conversely, since $a\sqrt{b} = \sqrt{a^2b}$,

A factor outside the radical sign may be raised to the corresponding power and placed under it.

Again:
$$\sqrt{\frac{a}{b^2}} = \sqrt{a \times \frac{1}{b^2}} = \frac{1}{b}\sqrt{a};$$

and
$$\sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}} = \sqrt{ab \times \frac{1}{b^2}} = \frac{1}{b}\sqrt{ab}.$$

270. A surd is in its *simplest form* when the expression under the radical sign is *integral* and *as small as possible.*

271. Surds which, when reduced to the simplest form, have the *same surd factor*, are said to be similar.

Simplify:

$$\sqrt{50};\quad \sqrt[3]{108};\quad \sqrt[5]{7x^2y^7};\quad \sqrt{\frac{7}{12}};\quad \sqrt[4]{\frac{5a}{2b^3c^2}};\quad \sqrt[3]{296352}.$$

(1) $\sqrt{50} = \sqrt{25\times 2} = 5\sqrt{2}.$

(2) $\sqrt[3]{108} = \sqrt[3]{27\times 4} = 3\sqrt[3]{4}.$

(3) $\sqrt[5]{7x^2y^7} = \sqrt[5]{7x^2y^2\times y^5} = y\sqrt[5]{7x^2y^2}.$

(4) $\sqrt{\frac{7}{12}} = \sqrt{\frac{7}{4\times 3}} = \sqrt{\frac{7\times 3}{4\times 9}} = \sqrt{21\times\frac{1}{4\times 9}} = \frac{1}{6}\sqrt{21}.$

(5) $\sqrt[4]{\frac{5a}{2b^3c^2}} = \sqrt[4]{\frac{40abc^2}{16b^4c^4}} = \frac{1}{2bc}\sqrt[4]{40abc^2}.$

(6) $\sqrt[3]{296352}.$

2^3	296352
2^2	37044
3^2	9261
3	1029
7	343
7	49
	7

Hence, $296352 = 2^5\times 3^3\times 7^3,$

$$\therefore \sqrt[3]{296352} = \sqrt[3]{7^3}\times\sqrt[3]{3^3}\times\sqrt[3]{2^5},$$
$$= 7\times 3\times 2\sqrt[3]{2^2},$$
$$= 42\sqrt[3]{4}.\ \textit{Ans.}$$

In simplifying numerical expressions under the radical sign, the method employed in (6) may be used with advantage when the factor whose root can be taken is not readily determined by inspection.

Exercise C.

Express as entire surds:

1. $3\sqrt{5};\ 3\sqrt{21};\ 5\sqrt{32};\ a^2b\sqrt{bc};\ x\sqrt[3]{x^2y^2}.$
2. $3y^2\sqrt[4]{x^3y};\ 2x\sqrt[5]{xy};\ a^3\sqrt[4]{a^3b^2};\ 3c^2\sqrt[3]{abc};\ 5abc\sqrt{abc^{-1}}.$
3. $\frac{2}{7}\sqrt{\frac{91}{8}};\ 16\sqrt{\frac{275}{288}};\ (x+y)\sqrt{\frac{xy}{x^2+2xy+y^2}}.$

Express as mixed surds:

4. $\sqrt{x^2y^4z}$; $\sqrt{8a^3b}$; $\sqrt[3]{54a^4x^2y^3}$; $\sqrt{24}$; $\sqrt{125a^4d^3}$.

5. $\sqrt[3]{1000a}$; $\sqrt[3]{160x^4y^7}$; $\sqrt[3]{108m^9n^{10}}$; $\sqrt[3]{1372a^{15}b^{16}}$.

6. $\sqrt[3]{a^4-3a^3b+3a^2b^2-ab^3}$; $\sqrt{50a^2-100ab+50b^2}$.

Simplify:

7. $2\sqrt[4]{80a^5b^2c^6}$; $7\sqrt{396x}$; $9\sqrt[3]{81x^2y^3z}$; $5\sqrt{726}$.

8. $\sqrt{\frac{5}{9}}$; $\sqrt{1\frac{11}{16}}$; $\sqrt{3\frac{1}{8}}$; $\frac{3}{5}\sqrt{90\frac{5}{8}}$; $2\sqrt[3]{\frac{1}{2}}$.

9. $\sqrt[3]{\dfrac{2xy^3}{z}}$; $\sqrt[3]{\frac{4}{25}}$; $\dfrac{a}{b}\sqrt[4]{\dfrac{b}{2a^3}}$; $\sqrt{\dfrac{3a^2bx}{4cy^3}}$.

10. $\dfrac{12}{\sqrt{5}}$; $\dfrac{2}{\sqrt{1701}}$; $\left(\dfrac{x^3y^2}{z^2}\right)\left(\dfrac{z^5}{x^5y^5}\right)^{\frac{1}{2}}$; $\left(\dfrac{a^3b^2}{c^4}\right)\left(\dfrac{c^9b^3}{a}\right)^{\frac{1}{3}}$.

11. $(ax)\times(b^2x)^{\frac{1}{2}}$; $(2a^2b^4)\times(b^2x^3)^{\frac{1}{3}}$; $5(3a^3b^4y)\times(a^5b^{-4}y^3)^{\frac{1}{4}}$.

12. Show that $\sqrt{20}$, $\sqrt{45}$, $\sqrt{\frac{4}{5}}$ are similar surds.

13. Show that $2\sqrt[3]{a^3b^2}$, $\sqrt[3]{8b^5}$, $\frac{1}{2}\sqrt[3]{\dfrac{a^6}{b}}$ are similar surds.

14. If $\sqrt{2}=1.414213$, find the values of

$$\sqrt{50};\quad \tfrac{5}{2}\sqrt{288};\quad \frac{1}{\sqrt{2}};\quad \frac{3}{\sqrt{450}}.$$

272. Surds of *the same order* may be compared by expressing them as entire surds.

Ex. Compare $\frac{2}{3}\sqrt{7}$ and $\frac{3}{5}\sqrt{10}$.

$$\tfrac{2}{3}\sqrt{7} = \sqrt{\tfrac{28}{9}},$$
$$\tfrac{3}{5}\sqrt{10} = \sqrt{\tfrac{18}{5}}.$$
$$\sqrt{\tfrac{28}{9}} = \sqrt{\tfrac{140}{45}}, \text{ and } \sqrt{\tfrac{18}{5}} = \sqrt{\tfrac{162}{45}}.$$

As $\sqrt{\frac{162}{45}}$ is greater than $\sqrt{\frac{140}{45}}$, $\frac{3}{5}\sqrt{10}$ is greater than $\frac{2}{3}\sqrt{7}$.

273. The product or quotient of two surds of *the same order* may be obtained by taking the product or quotient of the rational factors and the surd factors separately.

$$(1)\quad 2\sqrt{5}\times 5\sqrt{7}=10\sqrt{35}.$$

$$(2)\quad 9\sqrt{5}\div 3\sqrt{7}=3\sqrt{\tfrac{5}{7}}=3\sqrt{\tfrac{35}{49}}=\tfrac{3}{7}\sqrt{35}.$$

Exercise CI.

1. Which is the greater $3\sqrt{7}$ or $2\sqrt{15}$?
2. Arrange in order of magnitude $9\sqrt{3}$, $6\sqrt{7}$, $5\sqrt{10}$.
3. Arrange in order of magnitude $4\sqrt[3]{4}$, $3\sqrt[3]{5}$, $5\sqrt[3]{3}$.
4. Multiply $3\sqrt{2}$ by $4\sqrt{6}$; $\tfrac{2}{7}\sqrt{10}$ by $\tfrac{7}{10}\sqrt{15}$.
5. Multiply $5\sqrt{\tfrac{2}{7}}$ by $\tfrac{3}{7}\sqrt{162}$; $\tfrac{1}{2}\sqrt[3]{4}$ by $2\sqrt[3]{2}$.
6. Divide $2\sqrt{5}$ by $3\sqrt{15}$; $\tfrac{3}{8}\sqrt{21}$ by $\tfrac{9}{16}\sqrt{\tfrac{7}{20}}$.
7. Simplify $\tfrac{2}{5}\sqrt{3}\times\tfrac{4}{9}\sqrt{5}\div\tfrac{6}{7}\sqrt{2}$.
8. Simplify $\dfrac{2\sqrt{10}}{3\sqrt{27}}\times\dfrac{7\sqrt{48}}{5\sqrt{14}}\div\dfrac{4\sqrt{15}}{15\sqrt{21}}$.
9. Simplify $2\sqrt[3]{4}\times 5\sqrt[3]{32}\div\sqrt[3]{108}$.

274. The *order* of a surd may be changed by changing the *power* of the expression under the radical sign. Thus,

$$\sqrt{5}=\sqrt[4]{25};\quad \sqrt[3]{c}=\sqrt[6]{c^2}.$$

Conversely, $\quad \sqrt[4]{25}=\sqrt{5};\quad \sqrt[6]{c^2}=\sqrt[3]{c};$

or, in general, $\quad \sqrt[mn]{c^n}=\sqrt[m]{c}.$

In this way, surds of *different orders* may be reduced to the *same order*, and may then be compared, multiplied, or divided.

(1) To compare $\sqrt{2}$ and $\sqrt[3]{3}$.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{3}{6}} = \sqrt[6]{2^3} = \sqrt[6]{8};$$
$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{3^2} = \sqrt[6]{9}.$$

$\therefore\ \sqrt[3]{3}$ is greater than $\sqrt{2}$.

(2) To multiply $\sqrt[3]{4a}$ by $\sqrt{6x}$.

$$\sqrt[3]{4a} = (4a)^{\frac{1}{3}} = (4a)^{\frac{2}{6}} = \sqrt[6]{(4a)^2} = \sqrt[6]{16a^2};$$
$$\sqrt{6x} = (6x)^{\frac{1}{2}} = (6x)^{\frac{3}{6}} = \sqrt[6]{(6x)^3} = \sqrt[6]{216x^3}.$$

$$\begin{aligned}\therefore\ \sqrt[3]{4a} \times \sqrt{6x} &= \sqrt[6]{16a^2} \times \sqrt[6]{216x^3},\\ &= \sqrt[6]{16a^2 \times 216x^3},\\ &= \sqrt[6]{2^4a^2 \times 2^3 \times 3^3x^3},\\ &= \sqrt[6]{2^6 \times 2 \times 3^3a^2x^3},\\ &= 2\sqrt[6]{54a^2x^3}.\ \textit{Ans.}\end{aligned}$$

(3) To divide $\sqrt[3]{3a}$ by $\sqrt{6b}$.

$$\sqrt[3]{3a} = (3a)^{\frac{1}{3}} = (3a)^{\frac{2}{6}} = \sqrt[6]{(3a)^2} = \sqrt[6]{9a^2};$$
$$\sqrt{6b} = (6b)^{\frac{1}{2}} = (6b)^{\frac{3}{6}} = \sqrt[6]{(6b)^3} = \sqrt[6]{216b^3}.$$

$$\begin{aligned}\therefore\ \sqrt[3]{3a} \div \sqrt{6b} &= \sqrt[6]{9a^2} \div \sqrt[6]{216b^3} = \sqrt[6]{\frac{9a^2}{216b^3}},\\ &= \sqrt[6]{\frac{a^2}{24b^3}} = \sqrt[6]{\frac{a^2}{2^3 \times 3b^3}},\\ &= \sqrt[6]{\frac{2^3 \times 3^5a^2b^3}{2^6 \times 3^6b^6}} = \frac{1}{6b}\sqrt[6]{1944a^2b^3}.\ \textit{Ans.}\end{aligned}$$

Exercise CII.

Arrange in order of magnitude:

1. $2\sqrt[3]{3},\ 3\sqrt{2},\ \frac{5}{2}\sqrt[4]{4}$.
2. $\sqrt{\frac{3}{5}},\ \sqrt[3]{\frac{14}{15}}$.
3. $2\sqrt[3]{22},\ 3\sqrt[3]{7},\ 4\sqrt{2}$.
4. $3\sqrt{19},\ 5\sqrt[3]{2},\ 3\sqrt[3]{3}$.

Simpiify:

5. $2\sqrt{ax} \times \sqrt[3]{3a^2b} \times \sqrt{2bx}$; $\sqrt[4]{a^3xy^3} \times \sqrt[5]{a^2xy}$.

6. $3(4ab^2)^{\frac{1}{3}} \div (2a^3b)^{\frac{1}{2}}$; $(2a^3b^2)^{\frac{1}{4}} \times (a^5b^3)^{\frac{1}{3}} \div (a^3b^5)^{\frac{1}{2}}$.

7. $(2ab)^{\frac{1}{2}} \times (3ab^2)^{\frac{1}{3}} \div (5ab^3)^{\frac{1}{6}}$; $4\sqrt{12} \div 2\sqrt{3}$.

8. $\left(\dfrac{ay}{x}\right)^{\frac{1}{2}} \times \left(\dfrac{bx}{y^2}\right)^{\frac{1}{3}} \div \left(\dfrac{y^2}{a^2b^2}\right)^{\frac{1}{4}}$.

9. $(7\sqrt{2} - 5\sqrt{6} - 3\sqrt{8} + 4\sqrt{20}) \times 3\sqrt{2}$.

10. $\sqrt{(\frac{16}{25})^7} \times \sqrt{(\frac{25}{64})^6}$; $\sqrt[3]{(4ab^2)^x} \times \sqrt[3]{(2a^2b)^x}$.

11. $(\sqrt[7]{a^3b})^3 \times (\sqrt[7]{a^3b^{12}})^4$; $a^{\frac{1}{2}}b^{-\frac{2}{3}}c^{\frac{3}{4}}d^{-\frac{5}{6}} \div a^{\frac{7}{8}}b^{-\frac{9}{10}}c^{-\frac{10}{11}}d^{\frac{11}{12}}$.

275. In the addition or subtraction of surds, each surd must be reduced to its simplest form; and, if the resulting surds be similar,

Add the rational factors, and to their sum annex the common surd factor.

If the resulting surds be not similar,

Connect them with their proper signs.

276. Operations with surds will be more easily performed if the arithmetical numbers contained in the surds be *expressed in their prime factors*, and if *fractional exponents* be used instead of radical signs.

(1) Simplify $\sqrt{27} + \sqrt{48} + \sqrt{147}$.

$$\sqrt{27} = (3^3)^{\frac{1}{2}} = 3 \times 3^{\frac{1}{2}} = 3\sqrt{3};$$

$$\sqrt{48} = (2^4 \times 3)^{\frac{1}{2}} = 2^2 \times 3^{\frac{1}{2}} = 4 \times 3^{\frac{1}{2}} = 4\sqrt{3};$$

$$\sqrt{147} = (7^2 \times 3)^{\frac{1}{2}} = 7 \times 3^{\frac{1}{2}} = 7\sqrt{3}.$$

$$\therefore \sqrt{27} + \sqrt{48} + \sqrt{147} = (3 + 4 + 7)\sqrt{3} = 14\sqrt{3}. \text{ Ans.}$$

(2) Simplify $2\sqrt[3]{320} - 3\sqrt[3]{40}$.

$$2\sqrt[3]{320} = 2(2^6 \times 5)^{\frac{1}{3}} = 2 \times 2^2 \times 5^{\frac{1}{3}} = 8\sqrt[3]{5};$$
$$3\sqrt[3]{40} = 3(2^3 \times 5)^{\frac{1}{3}} = 3 \times 2 \times 5^{\frac{1}{3}} = 6\sqrt[3]{5}.$$
$$\therefore 2\sqrt[3]{320} - 3\sqrt[3]{40} = 8\sqrt[3]{5} - 6\sqrt[3]{5} = 2\sqrt[3]{5}. \textit{ Ans.}$$

(3) Find the square root of $\sqrt[3]{81}$.

The square root of $\sqrt[3]{81} = (81^{\frac{1}{3}})^{\frac{1}{2}} = 81^{\frac{1}{6}} = (3^4)^{\frac{1}{6}}$
$= 3^{\frac{2}{3}} = (3^2)^{\frac{1}{3}} = \sqrt[3]{9}.$

(4) Find the cube of $\frac{1}{2}\sqrt[6]{2}$.

The cube of $\frac{1}{2}\sqrt[6]{2} = (\frac{1}{2})^3 \times (2^{\frac{1}{6}})^3 = \frac{1}{8} \times 2^{\frac{1}{2}} = \frac{1}{8}\sqrt{2}.$

Exercise CIII.

Simplify:

1. $\sqrt{27} + 2\sqrt{48} + 3\sqrt{108}$; $3\sqrt{1000} + 4\sqrt{50} + 12\sqrt{288}$.
2. $\sqrt[3]{128} + \sqrt[3]{686} + \sqrt[3]{16}$; $7\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{432}$.
3. $12\sqrt{72} - 3\sqrt{128}$; $7\sqrt[3]{81} - 3\sqrt[3]{1029}$.
4. $2\sqrt{3} + 3\sqrt{1\frac{1}{3}} - \sqrt{5\frac{1}{3}}$; $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} - \sqrt{\frac{3}{5}}$.
5. $\sqrt{\dfrac{a^4c}{b^3}} - \sqrt{\dfrac{a^2c^3}{bd^2}} - \sqrt{\dfrac{a^2cd^2}{bm^2}}$; $3\sqrt{\frac{2}{5}} + 2\sqrt{\frac{1}{10}} - 4\sqrt{\frac{1}{40}}$.
6. $\sqrt{4a^3b} + \sqrt{25ab^3} - (a - 5b)\sqrt{ab}$.
7. $c\sqrt[5]{a^6b^7c^3} - a\sqrt[5]{ab^7c^8} + b\sqrt[5]{a^6b^2c^8}$.
8. $2\sqrt[3]{40} + 3\sqrt[3]{108} + \sqrt[3]{500} - \sqrt[3]{320} - 2\sqrt[3]{1372}$.
9. $(2\sqrt[6]{3a^4b})^3$; $(3\sqrt[6]{3})^2$.
10. $\left(\dfrac{a}{3}\sqrt{\dfrac{a}{3}}\right)^{\frac{1}{3}}$; $(\sqrt{27})^{\frac{1}{3}}$.
11. $(\sqrt[3]{81})^{\frac{1}{2}}$; $(\sqrt[4]{512})^{\frac{1}{3}}$; $(\sqrt[3]{256})^{-\frac{1}{4}}$; $\sqrt[12]{16}$; $\sqrt[12]{27}$.
12. $\sqrt[10]{4}$; $\sqrt[10]{36}$; $\sqrt[10]{32}$; $\sqrt[10]{243}$; $\sqrt[9]{125}$; $\sqrt[4]{49}$.
13. $\sqrt[9]{8x^6}$; $\sqrt[6]{9a^2b^4}$; $\sqrt[8]{16a^{12}}$; $\sqrt[5]{32a^{10}}$.

14. $(\sqrt[9]{8})^4$; $(\sqrt[9]{27})^4$; $(\sqrt[8]{64})^3$; $(\sqrt[3]{4})^2$.

15. $(a\sqrt[3]{a})^{-3}$; $(x\sqrt[3]{x})^{-\frac{1}{2}}$; $(p^2\sqrt{p})^{\frac{1}{5}}$; $(a^{-3}\sqrt[4]{a^{-3}})^{-\frac{1}{4}}$.

Expand by the method explained in § 201 :

16. $(\sqrt{a}+\sqrt{b})^5$; $(\sqrt[3]{m^2}+\sqrt{x^3})^3$; $(\sqrt{a}-2\sqrt{b})^5$.

17. $(2a^2-\frac{1}{2}\sqrt{a})^6$; $(2\sqrt[5]{x^4}-\frac{1}{2}y^2)^4$; $\left(\dfrac{2x^2}{y}-\sqrt[3]{y^2}\right)^6$.

18. $\left(\sqrt{ab}-\dfrac{c}{2\sqrt{b}}\right)^5$; $\left(\dfrac{a^2}{2c}-\dfrac{\sqrt{c}}{3}\right)^5$; $\left(a^2b-\dfrac{\sqrt{b}}{2a}\right)^4$.

19. $\left(\dfrac{a}{b}\sqrt{\dfrac{c}{d}}-\sqrt{\dfrac{d^2}{c^2}}\right)^3$; $(a^{\frac{n}{2}}-a^{-\frac{n}{2}})^4$; $\left(\dfrac{2a}{b^2}-\frac{1}{3}b\sqrt{a}\right)^4$.

20. $\left(\sqrt{\dfrac{a}{bc}}-\dfrac{\sqrt{c}}{3ab}\right)^3$; $\left(\dfrac{\sqrt{a}}{2\sqrt[3]{b^2}}-3\sqrt{b}\right)^3$; $\left(\dfrac{a\sqrt{a}}{\sqrt[6]{b^5}}-\dfrac{\sqrt[4]{b}}{2a}\right)^3$

Find the square root of:

21. $x^{4m}+6x^{3m}y^n+11x^{2m}y^{2n}+6x^my^{3n}+y^{4n}$.

22. $1+4x^{-\frac{1}{3}}-2x^{-\frac{2}{3}}-4x^{-1}+25x^{-\frac{4}{3}}-24x^{-\frac{5}{3}}+16x^{-2}$.

277. If we wish to find the approximate value of $\dfrac{3}{\sqrt{2}}$, it will be less labor to multiply first both numerator and denominator by a factor that will render the denominator *rational;* in this case by $\sqrt{2}$. Thus,

$$\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{\sqrt{2}\times\sqrt{2}}=\frac{3\sqrt{2}}{2}.$$

278. It is easy to rationalize the denominator of a fraction when that denominator is a *binomial* involving only quadratic surds. The factor required will consist of the same terms as the given denominator, but with a different

sign between them. Thus, $\frac{7-3\sqrt{5}}{6+2\sqrt{5}}$ will have its denominator rationalized by multiplying both terms of the fraction by $6-2\sqrt{5}$. For,

$$\frac{7-3\sqrt{5}}{6+2\sqrt{5}}=\frac{(7-3\sqrt{5})(6-2\sqrt{5})}{(6+2\sqrt{5})(6-2\sqrt{5})}$$

$$=\frac{72-32\sqrt{5}}{16}=\frac{9}{2}-2\sqrt{5}.$$

279. By two operations the denominator of a fraction may be rationalized when that denominator consists of *three* quadratic surds.

Thus, if the denominator be $\sqrt{6}+\sqrt{3}-\sqrt{2}$, both terms of the fraction may be multiplied by $\sqrt{6}-\sqrt{3}+\sqrt{2}$. The resulting denominator will be $6-5+2\sqrt{6}=1+2\sqrt{6}$; and if both terms of the resulting fraction be multiplied by $1-2\sqrt{6}$, the denominator will become $1-24=-23$.

Exercise CIV.

Find equivalent fractions with rational denominators, for the following:

1. $\frac{3}{\sqrt{7}+\sqrt{5}}$; $\frac{7}{2\sqrt{5}-\sqrt{6}}$; $\frac{4-\sqrt{2}}{1+\sqrt{2}}$; $\frac{6}{5-2\sqrt{6}}$.

2. $\frac{a}{\sqrt{b}-\sqrt{c}}$; $\frac{a+b}{a-\sqrt{b}}$; $\frac{2x-\sqrt{xy}}{\sqrt{xy}-2y}$.

Find the approximate values of:

3. $\frac{2}{\sqrt{3}}$; $\frac{1}{\sqrt{5}-\sqrt{2}}$; $\frac{7\sqrt{5}}{\sqrt{7}+\sqrt{3}}$; $\frac{7+2\sqrt{10}}{7-2\sqrt{10}}$.

IMAGINARY EXPRESSIONS.

280. All imaginary square roots may be reduced to one form.

$$\sqrt{-x^2} = \sqrt{x^2 \times (-1)} = x\sqrt{-1}.$$

$$\sqrt{-a} = \sqrt{a \times (-1)} = a^{\frac{1}{2}}\sqrt{-1}.$$

281. $\sqrt{-1}$ means an expression which, when multiplied by itself, produces -1. Therefore,

$$(\sqrt{-1})^2 = -1;$$

$$(\sqrt{-1})^3 = (\sqrt{-1})^2 \times \sqrt{-1} = -1\sqrt{-1} = -\sqrt{-1};$$

$$(\sqrt{-1})^4 = (\sqrt{-1})^2 \times (\sqrt{-1})^2 = (-1) \times (-1) = 1;$$

and so on. So that the successive powers of $\sqrt{-1}$ form the repeating series, $+\sqrt{-1}$, -1, $-\sqrt{-1}$, $+1$.

(1) Multiply $1 + \sqrt{-4}$ by $1 - \sqrt{-4}$.

$$1 + \sqrt{-4} = 1 + 2\sqrt{-1};$$

$$1 - \sqrt{-4} = 1 - 2\sqrt{-1}.$$

$$(1 + 2\sqrt{-1})(1 - 2\sqrt{-1}) = 1 - 4(-1) = 5.$$

(2) Divide $\sqrt{-ab}$ by $\sqrt{-b}$.

$$\sqrt{-ab} = a^{\frac{1}{2}}b^{\frac{1}{2}}\sqrt{-1},$$

and $$\sqrt{-b} = b^{\frac{1}{2}}\sqrt{-1}.$$

$$\frac{\sqrt{-ab}}{\sqrt{-b}} = \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}\sqrt{-1}}{b^{\frac{1}{2}}\sqrt{-1}} = \sqrt{a}.$$

EXERCISE CV.

Multiply:

1. $4 + \sqrt{-3}$ by $4 - \sqrt{-3}$; $\sqrt{3} - 2\sqrt{-2}$ by $\sqrt{3} + 2\sqrt{-2}$.
2. $\sqrt{54}$ by $\sqrt{-2}$; $a\sqrt{-b}$ by $x\sqrt{-y}$.

3. $\sqrt{-a}+\sqrt{-b}$ by $\sqrt{-a}-\sqrt{-b}$; $a\sqrt{-a^3b^4}$ by $\sqrt{-a^4b^5}$.

4. $\sqrt{-10}$ by $\sqrt{-2}$; $2\sqrt{3}-6\sqrt{-5}$ by $4\sqrt{3}-\sqrt{-5}$.

Divide:

5. $x\sqrt{-1}$ by $y\sqrt{-1}$; 1 by $\sqrt{-1}$; a by $a^{\frac{1}{2}}\sqrt{-1}$.

6. $\sqrt{-12}$ by $\sqrt{-3}$; $\sqrt{15}$ by $\sqrt{-5}$; $\sqrt{-5}$ by $\sqrt{-20}$.

Square Root of a Binomial Surd.

282. *The product or quotient of two dissimilar quadratic surds will be a quadratic surd.* Thus,

$$\sqrt{ab}\times\sqrt{abc}=ab\sqrt{c};$$
$$\sqrt{abc}\div\sqrt{ab}=\sqrt{c}.$$

For every quadratic surd, when simplified, will have under the radical sign one or more factors raised only to the first power; and two surds which are *dissimilar* cannot have *all* these factors alike.

Hence, their product or quotient will have *at least one factor* raised only to the *first* power, and will therefore be a surd.

283. *The sum or difference of two dissimilar quadratic surds cannot be a rational number, nor can it be expressed as a single surd.*

For if $\sqrt{a}\pm\sqrt{b}$ could equal a rational number c, we should have, by squaring,

$$a\pm 2\sqrt{ab}+b=c^2;$$

that is, $\pm 2\sqrt{ab}=c^2-a-b.$

Now, as the right side of this equation is rational, the left side would be rational; but, by § 282, $\sqrt{ab}$ cannot be rational. Therefore, $\sqrt{a}\pm\sqrt{b}$ cannot be rational.

In like manner, it may be shown that $\sqrt{a}\pm\sqrt{b}$ cannot be expressed as a single surd $\sqrt{c}$.

284. A *quadratic surd cannot equal the sum of a rational number and a surd.*

For, if $\sqrt{a}$ could equal $c+\sqrt{b}$, we should have, by squaring,

$$a=c^2+2c\sqrt{b}+b,$$

and, by transposing, $2c\sqrt{b}=a-b-c^2$.

That is, a surd equal to a rational number, which is impossible.

285. *If* $a+\sqrt{b}=x+\sqrt{y}$, *then* a *will equal* x *and* b *will equal* y.

For, by transposing, $\sqrt{b}-\sqrt{y}=x-a$; and if b were not equal to y, the difference of two unequal surds would be rational, which by § 283 is impossible.

$$\therefore b=y \text{ and } a=x.$$

In like manner, if $a-\sqrt{b}=x-\sqrt{y}$, a will equal x and b will equal y.

286. *To extract the square root of a binomial surd* $a+\sqrt{b}$.

Let $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$.

Squaring, $a+\sqrt{b}=x+2\sqrt{xy}+y$.

$\therefore x+y=a$, and $2\sqrt{xy}=\sqrt{b}$. 285.

From these two equations the values of x and y may be found.

This method may be shortened by observing that, since $\sqrt{b}=2\sqrt{xy}$,

$$a-\sqrt{b}=x-2\sqrt{xy}+y.$$

By taking the root, $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

$$\therefore (\sqrt{a+\sqrt{b}})(\sqrt{a-\sqrt{b}})=(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}).$$

$$\therefore \sqrt{a^2-b}=x-y.$$

And, as $a=x+y$,

the values of x and y may be found by addition and subtraction.

(1) Extract the square root of $7+4\sqrt{3}$.

$$\begin{aligned}
&\text{Let} && \sqrt{x}+\sqrt{y}=\sqrt{7+4\sqrt{3}}.\\
&\text{Then} && \sqrt{x}-\sqrt{y}=\sqrt{7-4\sqrt{3}}.\\
&\text{By multiplying,} && x-y=\sqrt{49-48},\\
& && \therefore x-y=1.\\
&\text{But} && x+y=7,\\
& && \therefore x=4, \text{ and } y=3.\\
& && \therefore \sqrt{x}+\sqrt{y}=2+\sqrt{3}.\\
& && \therefore \sqrt{7+4\sqrt{3}}=2+\sqrt{3}.
\end{aligned}$$

Exercise CVI.

Extract the square roots of:

1. $14+6\sqrt{5}$.
2. $17+4\sqrt{15}$.
3. $10+2\sqrt{21}$.
4. $16+2\sqrt{55}$.
5. $9-2\sqrt{14}$.
6. $20-8\sqrt{6}$.
7. $9-6\sqrt{2}$.
8. $94-42\sqrt{5}$.
9. $13-2\sqrt{30}$.
10. $11-6\sqrt{2}$.
11. $14-4\sqrt{6}$.
12. $38-12\sqrt{10}$.
13. $103-12\sqrt{11}$.
14. $57-12\sqrt{15}$.
15. $3\frac{1}{2}-\sqrt{10}$.
16. $2a+2\sqrt{a^2-b^2}$.
17. $a^2-2b\sqrt{a^2-b^2}$.
18. $87-12\sqrt{42}$.
19. $(a+b)^2-4(a-b)\sqrt{ab}$.

287. A root may often be obtained by inspection. For this purpose, write the given expression in the form $a+2\sqrt{b}$, and determine what two numbers have their sum equal a, and their product equal b.

(1) Find by inspection the square root of $18+2\sqrt{77}$.

It is required to find two numbers whose sum is 18 and whose product is 77; and these are evidently 11 and 7.

$$\begin{aligned}
&\text{Then} && 18+2\sqrt{77}=11+7+2\sqrt{11\times 7},\\
& && =(\sqrt{11}+\sqrt{7})^2.
\end{aligned}$$

That is, $\sqrt{11}+\sqrt{7}=$ square root of $18+2\sqrt{77}$.

(2) Find by inspection the square root of $75-12\sqrt{21}$.

It is necessary that the coefficient of the surd be 2; therefore, $75-12\sqrt{21}$ must be put in the form of

$$75-2\sqrt{756}.$$

The two numbers whose sum is 75 and whose product is 756 are 63 and 12.

Then $\quad 75-2\sqrt{756}=63+12-2\sqrt{63\times 12},$

$$=(\sqrt{63}-\sqrt{12})^2.$$

That is, $\quad \sqrt{63}-\sqrt{12}=$ square root of $75-12\sqrt{21}$;

or, $\quad 3\sqrt{7}-2\sqrt{3}=$ square root of $75-12\sqrt{21}$.

EQUATIONS CONTAINING RADICALS.

288. An equation containing a *single* radical may be solved by arranging the terms so as to have the radical alone on one side, and then raising both sides to a power corresponding to the order of the radical.

Ex. $\sqrt{x^2-9}+x=9$.

$$\sqrt{x^2-9}=9-x.$$

By squaring, $\quad x^2-9=81-18x+x^2.$

$$18x=90,$$

$$\therefore x=5.$$

289. If *two* radicals be involved, two steps may be necessary.

Ex. $\sqrt{x+15}+\sqrt{x}=15$.

$$\sqrt{x+15}+\sqrt{x}=15.$$

By squaring,

$$x+15+2\sqrt{x^2+15x}+x=225.$$

By transposing, $\quad 2\sqrt{x^2+15x}=210-2x.$

By dividing by 2, $\quad \sqrt{x^2+15x}=105-x.$

By squaring, $\quad x^2+15x=11025-210x+x^2.$

$$225x=11025,$$

$$\therefore x=49.$$

Some of the following radical equations will reduce to simple and others to quadratic equations.

Exercise CVII.

Solve:

1. $\sqrt{x-5}=2.$

2. $2\sqrt{3x+4}-x=4.$

3. $3-\sqrt{x^2-1}=2x.$

4. $\sqrt{3x-2}=2(x-4).$

5. $4x-12\sqrt{x}=16.$

6. $\sqrt{x+4}+\sqrt{2x-1}=6.$

7. $\sqrt{13x-1}-\sqrt{2x-1}=5.$

8. $\sqrt{4+x}+\sqrt{x}=3.$

9. $\sqrt{25+x}+\sqrt{25-x}=8.$

10. $x^2=21+\sqrt{x^2-9}.$

11. $2x-\sqrt[3]{8x^3+26}+2=0.$

12. $\sqrt{x+1}+\sqrt{x+16}=\sqrt{x+25}.$

13. $\sqrt{2x+1}-\sqrt{x+4}=\frac{1}{3}\sqrt{x-3}.$

14. $\sqrt{x+3}+\sqrt{x+8}=5\sqrt{x}.$

15. $\sqrt{3+x}+\sqrt{x}=\dfrac{6}{\sqrt{3+x}}.$

16. $\sqrt{x^2-1}+6=\dfrac{16}{\sqrt{x^2-1}}.$

17. $\dfrac{1}{\sqrt{x+1}}+\dfrac{1}{\sqrt{x-1}}=\dfrac{1}{\sqrt{x^2-1}}.$

18. $\dfrac{\sqrt{x+2a}-\sqrt{x-2a}}{\sqrt{x-2a}+\sqrt{x+2a}}=\dfrac{x}{2a}.$

19. $\dfrac{3x+\sqrt{4x-x^2}}{3x-\sqrt{4x-x^2}}=2.$

20. $\dfrac{\sqrt{7x^2+4}+2\sqrt{3x-1}}{\sqrt{7x^2+4}-2\sqrt{3x-1}}=7$

21. $\sqrt{(x-a)^2+2ab+b^2}=x-a+b.$

22. $\sqrt{(x+a)^2+2ab+b^2}=b-a-x.$

23. $\sqrt{\frac{x}{4}+3}+\sqrt{\frac{x}{4}-3}=\sqrt{\frac{2x}{3}}.$

24. $4x^{\frac{1}{2}} - 3(x^{\frac{1}{2}} + 1)(x^{\frac{1}{2}} - 2) = x^{\frac{1}{2}}(10 - 3x^{\frac{1}{2}})$.

25. $(x^{\frac{2}{3}} - 2)(x^{\frac{4}{3}} - 4) = x^{\frac{2}{3}}(x^{\frac{2}{3}} - 1)^2 - 12$.

26. $x^3 - 4x^{\frac{3}{2}} = 96$.

27. $x + x^{-1} = 2.9$.

28. $x^{\frac{1}{2}} + 2a^2x^{-\frac{1}{2}} = 3a$.

29. $81\sqrt[3]{x} + \dfrac{81}{\sqrt[3]{x}} = 52x$.

290. Equations may be solved with respect to an *expression* in the same manner as with respect to a letter.

(1) Solve $(x^2 - x)^2 - 8(x^2 - x) + 12 = 0$.

Consider $(x^2 - x)$ as the unknown quantity.

Then $\quad (x^2 - x)^2 - 8(x^2 - x) = -12$.

Complete the square, $(x^2 - x)^2 - (\ ) + 16 = 4$.

Extract the root, $\quad (x^2 - x) - 4 = \pm 2$.

$$x^2 - x = 6 \text{ or } 2.$$

Complete the square, $\quad 4x^2 - (\ ) + 1 = 25 \text{ or } 9$.

Extract the root, $\quad 2x - 1 = \pm 5 \text{ or } \pm 3$.

$$2x = 6,\ -4,\ 4,\ -2.$$

$$\therefore x = 3,\ -2,\ 2,\ -1.$$

(2) Solve $5x - 7x^2 - 8\sqrt{7x^2 - 5x + 1} = 8$.

Change the signs and annex $+1$ to both sides.

$$7x^2 - 5x + 1 + 8\sqrt{7x^2 - 5x + 1} = -7.$$

Solve with respect to $\sqrt{7x^2 - 5x + 1}$.

$$(7x^2 - 5x + 1) + 8(7x^2 - 5x + 1)^{\frac{1}{2}} + 16 = 9.$$

$$(7x^2 - 5x + 1)^{\frac{1}{2}} + 4 = \pm 3.$$

$$(7x^2 - 5x + 1)^{\frac{1}{2}} = -1 \text{ or } -7.$$

Square, $\quad 7x^2 - 5x + 1 = 1 \text{ or } 49$.

Transpose, $\quad 7x^2 - 5x = 0 \text{ or } 48$.

From $7x^2 - 5x = 0$, $\quad x = 0 \text{ or } \frac{5}{7}$;

From $7x^2 - 5x = 48$, $\quad x = 3 \text{ or } -2\frac{2}{7}$.

NOTE. In verifying the values of x in the original equation, it is seen that the value of $\sqrt{7x^2 - 5x + 1}$ is negative. Thus, by putting 0 for x the equation becomes $0 - 8\sqrt{1} = 8$; and by taking -1 for $\sqrt{1}$ we have $(-8)(-1) = 8$; that is, $8 = 8$.

(3) Solve $x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=1.$

Arrange as follows: $\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=0.$

By adding 2 to $\left(x^2+\frac{1}{x^2}\right)$,

there is obtained $\quad x^2+2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2.$

$$\therefore \left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=2.$$

Multiply by 4 and complete the square,

$$4\left(x+\frac{1}{x}\right)^2+(\;)+1=9.$$

Extract the root, $\quad 2\left(x+\frac{1}{x}\right)+1=\pm 3.$

$$2\left(x+\frac{1}{x}\right)=2 \text{ or } -4.$$

$$x+\frac{1}{x}=1 \text{ or } -2.$$

Multiply by x, $\quad x^2-x=-1, \qquad$ and $\qquad x^2+2x=-1.$

$$\therefore 4x^2-(\;)+1=-3, \qquad \therefore x^2+2x+1=0.$$

$$2x-1=\pm\sqrt{-3}, \qquad x+1=0.$$

$$\therefore x=\tfrac{1}{2}(1\pm\sqrt{-3}). \qquad \therefore x=-1.$$

291. An equation like that of (3) which will remain unaltered when $\frac{1}{x}$ is substituted for x, is called a **reciprocal equation.**

It will be found that every reciprocal equation of *odd* degree will be divisible by $x-1$ or $x+1$ according as the last term is negative or positive; and every reciprocal equation of *even* degree *with its last term negative* will be divisible by x^2-1. In every case the equation resulting from the division will be reciprocal.

(4) Solve $x^5 + 2x^4 - 3x^3 - 3x^2 + 2x + 1 = 0.$

This is a reciprocal equation, for, if x^{-1} be put for x, the equation becomes $x^{-5} + 2x^{-4} - 3x^{-3} - 3x^{-2} + 2x^{-1} + 1 = 0$, which multiplied by x^5 gives $1 + 2x - 3x^2 - 3x^3 + 2x^4 + x^5 = 0$, the same as the original equation.

The equation may be written $(x^5+1) + 2x(x^3+1) - 3x^2(x+1) = 0$, which is obviously divisible by $x+1$. The result from dividing by $x+1$ is $x^4 + x^3 - 4x^2 + x + 1 = 0$, or $(x^4+1) + x(x^2+1) = 4x^2$. By adding $2x^2$ to (x^4+1) it becomes $(x^4 + 2x^2 + 1) = (x^2+1)^2$.

Then $$(x^2+1)^2 + x(x^2+1) = 6x^2.$$

Multiply by 4 and complete the square,

$$4(x^2+1)^2 + (\ ) + x^2 = 25x^2.$$

eight times, 10 five times, and four units.

293. In this system a number is represented by a series of *different* powers of 10, the exponent of each power being *integral*. But, by employing *fractional* exponents, any number may be represented (approximately) as a *single* power of 10.

294. When numbers are referred in this way to 10, the **exponents of the powers** corresponding to them are called their **logarithms** to the base 10.

For brevity the word "logarithm" is written log.

From § 255 it appears that:

$$10^0 = 1, \qquad 10^{-1}\left(=\tfrac{1}{10}\right) = .1,$$
$$10^1 = 10, \qquad 10^{-2}\left(=\tfrac{1}{100}\right) = .01,$$
$$10^2 = 100, \qquad 10^{-3}\left(=\tfrac{1}{1000}\right) = .001,$$

5. $3(2x^2 - x) - ($ …

6. $15x - 3x^2 + 4(x^2 - 5x + 5)^{\frac{1}{2}} = 16.$

7. $x^2 + x^{-2} + x + x^{-1} = 4.$

8. $x^2 + \sqrt{x^2 - 7} = 19.$

9. $x^2 + x + \frac{1}{6}(x^2 + x)^{\frac{1}{2}} = \frac{7}{6}.$

10. $(x+1)^{\frac{1}{2}} + (x-1)^{\frac{1}{2}} = 5.$

(3) Solve $x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=1.$

Arrange as follows: $\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=0.$

By adding 2 to $\left(x^2+\frac{1}{x^2}\right)$,

there is obtained $\quad x^2+2+\frac{1}{x^2}=\left(x+\frac{1}{x}\right)^2.$

$$\therefore \left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)=2.$$

Multiply by 4 and complete the square,

$$4\left(x+\frac{1}{x}\right)^2+(\ )+1=9.$$

Extract the root, $2\left(x+\frac{1}{x}\right)+1=\pm 3.$

$$2\left(x+\frac{1}{x}\right)=2 \text{ or } -4.$$

$$x+\frac{1}{x}=1 \text{ or } -2.$$

Multiply by x, $\quad x^2-x=-1, \qquad$ and $\qquad x^2+2x=-1.$

$$\therefore 4x^2-(\ )+1=-3, \qquad \therefore x^2+2x+1=0.$$

$$2x-1=\pm\sqrt{-3}, \qquad x+1=0.$$

$$\therefore x=\tfrac{1}{2}(1\pm\sqrt{-3}). \qquad \therefore x=-1.$$

291. An equation like that of (3) which will remain unaltered when $\frac{1}{x}$ is substituted for x, is called a **reciprocal equation.**

28. $\frac{x^2}{a}-3ax=\sqrt{4x^3+9ax^2}+\frac{27a^2}{4}.$

29. $(x+1+x^{-1})(x-1+x^{-1})=5\tfrac{1}{4}.$

30. $2(x^{\frac{1}{2}}-1)^{-1}-2(x^{\frac{1}{2}}-4)^{-1}=3(x^{\frac{1}{2}}-2)^{-1}.$

(4) Solve $x^5+2x^4-3x^3-3x^2+2x+1=0$.

This is a reciprocal equation, for, if x^{-1} be put for x, the equation becomes $x^{-5}+2x^{-4}-3x^{-3}-3x^{-2}+2x^{-1}+1=0$, which multiplied by x^5 gives $1+2x-3x^2-3x^3+2x^4+x^5=0$, the same as the original equation.

The equation may be written $(x^5+1)+2x(x^3+1)-3x^2(x+1)=0$, which is obviously divisible by $x+1$. The result from dividing by $x+1$ is $x^4+x^3-4x^2+x+1=0$, or $(x^4+1)+x(x^2+1)=4x^2$. By adding $2x^2$ to (x^4+1) it becomes $(x^4+2x^2+1)=(x^2+1)^2$.

Then $(x^2+1)^2+x(x^2+1)=6x^2$.

Multiply by 4 and complete the square,

$4(x^2+1)^2+(\ )+x^2=25x^2$.

eight times, 10 five times, and four units.

293. In this system a number is represented by a series of *different* powers of 10, the exponent of each power being *integral*. But, by employing *fractional* exponents, any number may be represented (approximately) as a *single* power of 10.

294. When numbers are referred in this way to 10, the **exponents of the powers** corresponding to them are called their **logarithms** to the base 10.

For brevity the word "logarithm" is written log.

From § 255 it appears that:

$$10^0 = 1, \quad 10^{-1}\left(=\tfrac{1}{10}\right) = .1,$$
$$10^1 = 10, \quad 10^{-2}\left(=\tfrac{1}{100}\right) = .01,$$
$$10^2 = 100, \quad 10^{-3}\left(=\tfrac{1}{1000}\right) = .001,$$

and so on. Hence,

$$\log 1 = 0, \quad \log .1 = -1,$$
$$\log 10 = 1, \quad \log .01 = -2,$$
$$\log 100 = 2, \quad \log .001 = -3,$$

and so on.

It is evident that the logarithms of all numbers between

1 and	10 will be	$0 +$ a fraction,
10 and	100 will be	$1 +$ a fraction,
100 and	1000 will be	$2 +$ a fraction,
1 and	.1 will be	$-1 +$ a fraction,
.1 and	.01 will be	$-2 +$ a fraction,
.01 and	.001 will be	$-3 +$ a fraction.

295. The fractional part of a logarithm cannot be expressed *exactly* either by common or by decimal fractions; but decimals may be obtained for these fractional parts, true to as many places as may be desired.

If, for instance, the logarithm of 2 be required; log 2 may be supposed to be $\frac{1}{3}$.

Then $10^{\frac{1}{3}} = 2$; or, by raising both sides to the *third* power, $10 = 8$, a result which shows that $\frac{1}{3}$ is too large.

Suppose, then, $\log 2 = \frac{3}{10}$. Then $10^{\frac{3}{10}} = 2$, or by raising both sides to the *tenth* power, $10^3 = 2^{10}$. That is, $1000 = 1024$, a result which shows that $\frac{3}{10}$ is too small.

Since $\frac{1}{3}$ is too large and $\frac{3}{10}$ too small, log 2 lies between $\frac{1}{3}$ and $\frac{3}{10}$; that is, between .33333 and .30000.

In supposing log 2 to be $\frac{1}{3}$, the error of the result is $\frac{10-8}{10} = \frac{2}{10} = .2$. In supposing log 2 to be $\frac{3}{10}$, the error of the result is $\frac{1000-1024}{1000} = \frac{-24}{1000} = -.024$; log 2, therefore, is nearer to $\frac{3}{10}$ than to $\frac{1}{3}$.

The difference between the errors is $.2-(-.024) = .224$, and the difference between the supposed logarithms is $.33333 - .3 = .03333$.

The last error, therefore, in the supposed logarithm may be considered to be approximately $\frac{24}{224}$ of $.03333 = .0035$ nearly, and this added to .3000 gives .3035, a result a little too large.

By shorter methods of higher mathematics, the logarithm of 2 is known to be 0.3010300, true to the seventh place.

296. The logarithm of a number consists of two parts, an integral part and a fractional part.

Thus, $\log 2 = 0.30103$, in which the integral part is 0, and the fractional part is .30103; $\log 20 = 1.30103$, in which the integral part is 1, and the fractional part is .30103.

297. The integral part of a logarithm is called the **characteristic**; and the fractional part is called the **mantissa.**

298. The mantissa is always made *positive.* Hence, in the case of numbers less than 1 whose logarithms are *negative,* the logarithm is made to consist of a *negative* characteristic and a *positive* mantissa.

299. When a logarithm consists of a *negative* characteristic and a *positive* mantissa, it is usual to write the minus sign *over* the characteristic, or else to add 10 to the characteristic and to indicate the subtraction of 10 from the resulting logarithm.

Thus, $\log .2 = \bar{1}.30103$, and this may be written $9.30103 - 10$.

300. *The characteristic of a logarithm of an integral number, or of a mixed number, is one less than the number of integral digits.*

Thus, from § 294, $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$. Hence, the logarithms of all numbers from 1 to 10 (that is, of all numbers consisting of *one* integral digit), will have 0 for characteristic; and the logarithms of all numbers from 10 to 100 (that is, of all numbers consisting of *two* integral digits), will have 1 for characteristic; and so on, the characteristic increasing by 1 for each increase in the number of digits, and therefore always being 1 less than that number.

301. *The characteristic of a logarithm of a decimal fraction is negative, and is equal to the number of the place occupied by the first significant figure of the decimal.*

Thus, from § 294, $\log .1 = -1$, $\log .01 = -2$, $\log .001 = -3$. Hence, the logarithms of all numbers from .1 to 1 will have -1 for a characteristic (the mantissa being *plus*); the logarithms of all numbers from .01 to .1 will have -2 for a characteristic; the logarithms of all numbers from .01 to .001 will have -3 for a characteristic; and so on, the characteristic always being *negative and equal to the number of the place occupied by the first significant figure of the decimal.*

302. *The mantissa of a logarithm of any integral number or decimal fraction depends only upon the digits of the number, and is unchanged so long as the sequence of the digits remains the same.*

For, changing the position of the decimal point in a number is equivalent to multiplying or dividing the number by a power of 10. Its logarithm, therefore, will be increased or diminished by the *exponent* of that power of 10; and, since this exponent is *integral*, the *mantissa* of the logrithm will be unaffected.

Thus, if $27196 = 10^{4.4345}$,

then
$$2719.6 = 10^{3.4345},$$
$$27.196 = 10^{1.4345},$$
$$2.7196 = 10^{0.4345},$$
$$.27196 = 10^{9.4345-10},$$
$$.0027196 = 10^{7.4345-10}.$$

303. The advantage of using the number 10 as the base of a system of logarithms consists in the fact that the **mantissa** depends only on the **sequence of digits**, and the **characteristic** on the **position of the decimal point.**

304. As logarithms are simply exponents (§ 294), therefore,

The logarithm of a product is the sum of the logarithms of the factors.

Thus, $\log 20 = \log (2 \times 10) = \log 2 + \log 10,$
$$= 0.3010 + 1.0000 = 1.3010;$$
$$\log 2000 = \log (2 \times 1000) = \log 2 + \log 1000,$$
$$= 0.3010 + 3.0000 = 3.3010;$$
$$\log .2 = \log (2 \times .1) = \log 2 + \log .1,$$
$$= 0.3010 + 9.0000 - 10 = 9.3010 - 10;$$
$$\log .02 = \log (2 \times .01) = \log 2 + \log .01,$$
$$= 0.3010 + 8.0000 - 10 = 8.3010 - 10.$$

Exercise CIX.

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$.

Find the logarithms of the following numbers by resolv-

ing the numbers into factors, and taking the sum of the logarithms of the factors:

1. log 6.	9. log 25.	17. log .021.	25. log 2.1.
2. log 15.	10. log 30.	18. log .35.	26. log 16.
3. log 21.	11. log 42.	19. log .0035.	27. log .056.
4. log 14.	12. log 420.	20. log .004.	28. log .63.
5. log 35.	13. log 12.	21. log .05.	29. log 1.75.
6. log 9.	14. log 60.	22. log 12.5.	30. log 105.
7. log 8.	15. log 75.	23. log 1.25.	31. log .0105.
8. log 49.	16. log 7.5.	24. log 37.5.	32. log 1.05.

305. As logarithms are simply exponents (§ 294), therefore,

The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

Thus, $\log 5^7 = 7 \times \log 5 = 7 \times 0.6990 = 4.8930.$
$\log 3^{11} = 11 \times \log 3 = 11 \times 0.4771 = 5.2481.$

306. As logarithms are simply exponents (§ 294), therefore, when roots are expressed by fractional indices,

The logarithm of a root of a number is equal to the logarithm of the number multiplied by the index of the root.

Thus, $\log 2^{\frac{1}{4}} = \frac{1}{4}$ of $\log 2 = \frac{1}{4} \times 0.3010 = 0.0753.$
$\log .002^{\frac{1}{3}} = \frac{1}{3}$ of $(7.3010 - 10).$

The expression $\frac{1}{3}$ of $(7.3010 - 10)$
may be put in the form of $\frac{1}{3}$ of $(27.3010 - 30)$ which $= 9.1003 - 10$; for, since $20 - 20 = 0$, the addition of 20 to the 7, and of -20 to the -10, produces no change in the *value* of the logarithm.

307. *In simplifying the logarithm of a root the equal positive and negative numbers to be added to the logarithm must be such that the resulting negative number, when divided by the index of the root, shall give a quotient of* -10.

Exercise CX.

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$.

Find logarithms of the following:

1. 2^8.	6. 5^5.	11. $5^{\frac{1}{5}}$.	16. $7^{\frac{2}{7}}$.	21. $5^{\frac{7}{2}}$.
2. 5^2.	7. $2^{\frac{1}{3}}$.	12. $7^{\frac{1}{11}}$.	17. $5^{\frac{5}{3}}$.	22. $2^{\frac{11}{7}}$.
3. 7^4.	8. $5^{\frac{1}{2}}$.	13. $2^{\frac{3}{4}}$.	18. $3^{\frac{2}{11}}$.	23. $5^{\frac{3}{4}}$.
4. 3^8.	9. $3^{\frac{1}{5}}$.	14. $5^{\frac{2}{3}}$.	19. $7^{\frac{7}{2}}$.	24. $7^{\frac{11}{7}}$.
5. 7^3.	10. $7^{\frac{1}{5}}$.	15. $3^{\frac{2}{7}}$.	20. $3^{\frac{4}{5}}$.	25. $21^{\frac{7}{8}}$.

308. Since logarithms are simply exponents (§ 294), therefore,

The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.

Thus, $\log \frac{3}{2} = \log 3 - \log 2 = 0.4771 - 0.3010 = \quad 0.1761.$
$\log \frac{2}{3} = \log 2 - \log 3 = 0.3010 - 0.4771 = -0.1761.$

To avoid the negative logarithm -0.1761, we subtract the *entire* logarithm 0.1761 from 10, and then indicate the subtraction of 10 from the result.

Thus, $-0.1761 = 9.8239 - 10.$
Hence, $\log \frac{2}{3} = 9.8239 - 10.$

309. The remainder obtained by subtracting the logarithm of a number from 10 is called the **cologarithm** of the number, or **arithmetical complement** of the logarithm of the number.

Cologarithm is usually denoted by colog, and is most easily found by *beginning with the characteristic of the logarithm and subtracting each figure from* 9 *down to the last significant figure, and subtracting that figure from* 10.

Thus, $\log 7 = 0.8451$; and colog $7 = 9.1549$. Colog 7 is readily found by subtracting, mentally, 0 from 9, 8 from 9, 4 from 9, 5 from 9, 1 from 10, and writing the resulting figure at each step.

310. Since $\qquad \text{colog } 7 = 9.1549,$

and $\quad \log \frac{1}{7} = \log 1 - \log 7 = 0 - 0.8451 = 9.1549 - 10,$

it is evident that,

If 10 *be subtracted from the cologarithm of a number, the result is the logarithm of the reciprocal of that number.*

311. Since
$$\log \tfrac{7}{5} = \log 7 - \log 5,$$
$$= 0.8451 - 0.6990 = 0.1461,$$
and
$$\log 7 + \text{colog } 5 - 10 = 0.8451 + 9.3010 - 10,$$
$$= 0.1461,$$
it is evident that,

The addition of a cologarithm − 10 *is equivalent to the* **subtraction of a logarithm.**

• The steps that lead to this result are:

$$\tfrac{7}{5} = 7 \times \tfrac{1}{5},$$

therefore, $\quad \log \tfrac{7}{5} = \log (7 \times \tfrac{1}{5}) = \log 7 + \log \tfrac{1}{5}.$ § 3C4.

But $\quad \log \tfrac{1}{5} = \text{colog } 5 - 10.$ § 309.

Hence, $\quad \log \tfrac{7}{5} = \log 7 + \text{colog } 5 - 10.$

Therefore,

312. *The logarithm of a quotient may be found by adding together the logarithm of the dividend and the cologarithm of the divisor, and subtracting* 10 *from the result.*

In finding a cologarithm when the *characteristic* of the logarithm is a *negative* number, it must be observed that the *subtraction* of a *negative* number is equivalent to the *addition* of an *equal positive* number.

Thus,
$$\log \frac{5}{.002} = \log 5 + \text{colog } .002 - 10,$$
$$= 0.6990 + 12.6990 - 10,$$
$$= 3.3980.$$

Here $\log .002 = \bar{3}.3010$, and in subtracting -3 from 9 the result is the same as adding $+3$ to 9.

Again
$$\log \frac{2}{.07} = \log 2 + \text{colog } .07 - 10,$$
$$= 0.3010 + 11.1549 - 10,$$
$$= 1.4559.$$

Also, $\log \frac{.07}{2^3} = 8.8451 - 10 + 9.0970 - 10,$

$= 17.9421 - 20,$

$= 7.9421 - 10.$

Here, $\log 2^3 = 3 \log 2 = 3 \times 0.3010 = 0.9030.$

Hence, $\text{colog } 2^3 = 10 - 0.9030 = 9.0970.$

Exercise CXI.

Given: $\log 2 = 0.3010$; $\log 3 = 0.4771$; $\log 5 = 0.6990$; $\log 7 = 0.8451$.

Find logarithms for the following quotients:

1. $\frac{2}{5}$.	7. $\frac{5}{3}$.	13. $\frac{.05}{3}$.	19. $\frac{.05}{.003}$.	25. $\frac{.02^2}{3^3}$.
2. $\frac{2}{7}$.	8. $\frac{5}{2}$.	14. $\frac{.005}{2}$.	20. $\frac{.007}{.02}$.	26. $\frac{3^3}{.02^2}$.
3. $\frac{3}{5}$.	9. $\frac{7}{3}$.	15. $\frac{.07}{5}$.	21. $\frac{.02}{.007}$.	27. $\frac{7^3}{.02^2}$.
4. $\frac{3}{7}$.	10. $\frac{7}{2}$.	16. $\frac{5}{.07}$.	22. $\frac{.005}{.07}$.	28. $\frac{.07^3}{.003^3}$.
5. $\frac{5}{7}$.	11. $\frac{3}{2}$.	17. $\frac{3}{.007}$.	23. $\frac{.03}{7}$.	29. $\frac{.005^2}{7^3}$.
6. $\frac{7}{5}$.	12. $\frac{7}{.5}$.	18. $\frac{.003}{7}$.	24. $\frac{.0007}{.2}$.	30. $\frac{7^3}{.005^2}$.

313. A table of *four-place* logarithms is here given, which contains logarithms of all numbers under 1000, *the decimal point and characteristic being omitted.* The logarithms of single digits 1, 8, etc., will be found at 10, 80, etc.

Tables containing logarithms of more places can be procured, but this table will serve for many practical uses, and will enable the student to use tables of six-place, seven-place, and ten-place logarithms, in work that requires greater accuracy.

314. In working with a four-place table, the numbers corresponding to the logarithms, that is, the *antilogarithms*, as they are called, may be carried to *four significant digits*.

TO FIND THE LOGARITHM OF A NUMBER IN THIS TABLE.

315. Suppose it is required to find the logarithm of 65.7. In the column headed "N" look for the first two significant figures, and at the top of the table for the third significant figure. In the line with 65, and in the column headed 7, is seen 8176. To this number prefix the characteristic and insert the decimal point. Thus,

$$\log 65.7 = 1.8176.$$

Suppose it is required to find the logarithm of 20347. In the line with 20, and in the column headed 3, is seen 3075; also in the line with 20, and in the 4 column, is seen 3096, and the difference between these two is 21. The difference between 20300 and 20400 is 100, and the difference between 20300 and 20347 is 47. Hence, $\frac{47}{100}$ of $21 = 10$, nearly, must be added to 3075. That is,

$$\log 20347 = 4.3085.$$

Suppose it is required to find the logarithm of .0005076. In the line with 50, and in the 7 column, is seen 7050; in the 8 column, 7059: the difference is 9. The difference between 5070 and 5080 is 10, and the difference between 5070 and 5076 is 6. Hence, $\frac{6}{10}$ of $9 = 5$ must be added to 7050. That is,

$$\log .0005076 = 6.7055 - 10.$$

TO FIND A NUMBER WHEN ITS LOGARITHM IS GIVEN.

316. Suppose it is required to find the number of which the logarithm is 1.9736.

Look for 9736 in the table. In the column headed "N," and in the line with 9736, is seen 94, and at the head of

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

the column in which 9736 stands is seen 1. Therefore, write 941, and insert the decimal point as the characteristic directs. That is, the number required is 94.1.

Suppose it is required to find the number of which the logarithm is 3.7936.

Look for 7936 in the table. It cannot be found, but the two adjacent mantissas between which it lies are seen to be 7931 and 7938; their difference is 7, and the difference between 7931 and 7936 is 5. Therefore, $\frac{5}{7}$ of the difference between the numbers corresponding to the mantissas, 7931 and 7938, must be added to the number corresponding to the mantissa 7931.

The number corresponding to the mantissa 7938 is 6220.

The number corresponding to the mantissa 7931 is 6210.

The difference between these numbers is 10,

and $$6210 + \tfrac{5}{7} \text{ of } 10 = 6217.$$

Therefore, the number required is 6217.

Suppose it is required to find the number of which the logarithm is $7.3882 - 10$.

Look for 3882 in the table. It cannot be found, but the two adjacent mantissas between which it lies are seen to be 3874 and 3892; their difference is 18, and the difference between 3874 and 3882 is 8. Therefore, $\frac{8}{18}$ of the difference between the numbers corresponding to the mantissas, 3874 and 3892, must be added to the number corresponding to the mantissa 3874.

The number corresponding to the mantissa 3892 is 2450.

The number corresponding to the mantissa 3874 is 2440.

The difference between these numbers is 10,

and $$2440 + \tfrac{4}{9} \text{ of } 10 = 2444.$$

Therefore, the number required is .002444.

Exercise CXII.

Find logarithms of the following numbers:

1. 60.	6. 3780.	11. 70633.	16. 877.08.
2. 101.	7. 54327.	12. 12028.	17. 73.896.
3. 999.	8. 90801.	13. 0.00987.	18. 7.0699.
4. 9901.	9. 10001.	14. 0.87701.	19. 0.0897.
5. 5406.	10. 10010.	15. 1.0001.	20. 99.778.

Find antilogarithms to the following logarithms:

21. 4.2488.	25. 4.7317.	29. 9.0410 − 10.
22. 3.6330.	26. 1.9730.	30. 9.8420 − 10.
23. 2.5310.	27. 9.8800 − 10.	31. 7.0216 − 10.
24. 1.9484.	28. 0.2787.	32. 8.6580 − 10.

Ex. Find the product of $908.4 \times .05392 \times 2.117$.

$$\begin{aligned} \log\ 908.4 &= 2.9583 \\ \log\ .05392 &= 8.7318 - 10 \\ \log\ 2.117 &= \underline{0.3257\qquad\quad} \\ & \ 2.0158 = \log 103.7.\ \textit{Ans.} \end{aligned}$$

Find by logarithms the following products:

33. 948.76×0.043875.	35. 830.75×0.0003769.
34. 3.4097×0.0087634.	36. 8.4395×0.98274.

317. When any of the factors are *negative*, find their logarithms without regard to the signs; write the letter n after the logarithm that corresponds to a negative number. If the number of logarithms so marked be *odd*, the product is *negative;* if *even*, the product is *positive*.

Find the products of:

37. $7564 \times (-0.003764)$.
38. $3.7648 \times (-0.083497)$.
39. $-5.840359 \times (-0.00178)$.
40. -8945.07×73.846.

Ex. Find the quotient of $\dfrac{8.3709 \times 834.637}{7308.946}$.

$$\begin{array}{lrl} \log & 8.3709 & = 0.9227 \\ \log & 834.637 & = 2.9215 \\ \text{colog} & 7308.946 & = \underline{6.1362-10} \\ & & \ 9.9804-10 = \log .9558. \textit{ Ans.} \end{array}$$

Find the quotients of:

41. $\dfrac{70654}{54013}$.
42. $\dfrac{58706}{93078}$.
43. $\dfrac{8.32165}{0.07891}$.
44. $\dfrac{65039}{90761}$.
45. $\dfrac{7.652}{-0.06875}$.
46. $\dfrac{0.07654}{83.947 \times 0.8395}$.
47. $\dfrac{7564 \times 0.07643}{8093 \times 0.09817}$.
48. $\dfrac{89 \times 753 \times 0.0097}{36709 \times 0.08497}$.
49. $\dfrac{413 \times 8.17 \times 3182}{915 \times 728 \times 2.315}$.
50. $\dfrac{212 \times (-6.12) \times (-2008)}{365 \times (-531) \times 2.576}$.

Ex. Find the cube of .0497.

$$\begin{array}{rl} \log .0497 = & 8.6964-10 \\ & \underline{3} \\ & 6.0892-10 = \log .0001228. \textit{ Ans.} \end{array}$$

Find by logarithms:

51. 6.05^3.
52. 1.051^7.
53. 1.1768^5.
54. 1.3178^{10}.
55. 0.78765^6.
56. 0.691^9.
57. $(\frac{73}{81})^{11}$.
58. $(\frac{14}{51})^7$.
59. $(10\frac{2}{3})^4$.
60. $(1\frac{7}{9})^8$.
61. $(\frac{951}{823})^6$.
62. $(7\frac{6}{11})^{0.38}$.
63. $(3\frac{27}{81})^{4.17}$.
64. $(1\frac{2}{11})^{3.2}$.
65. $(8\frac{3}{4})^{2.3}$.
66. $(5\frac{31}{37})^{0.375}$.

Ex. Find the fourth root of 0.00862.

$$\begin{array}{rl} \log 0.00862 = & 7.9355 - 10 \\ & \underline{30. \qquad - 30} \\ 4 \overline{)} & 37.9355 - 40 \\ & 9.4839 - 10 = \log .3047. \quad \textit{Ans.} \end{array}$$

Find by logarithms:

67. $7^{\frac{1}{3}}$.

68. $11^{\frac{1}{5}}$.

69. $783^{\frac{1}{3}}$.

70. $8379^{\frac{1}{10}}$.

71. $906.80^{\frac{1}{4}}$.

72. $8.1904^{\frac{1}{5}}$.

73. $0.17643^{\frac{5}{6}}$.

74. $2.5637^{\frac{3}{11}}$.

75. $\left(\frac{481}{788}\right)^{\frac{1}{2}}$.

76. $\left(\frac{71}{48406}\right)^{\frac{4}{7}}$.

77. $\left(9\frac{21}{48}\right)^{\frac{1}{5}}$.

78. $\left(11\frac{21}{71}\right)^{\frac{4}{5}}$.

Find by logarithms the values of:

79. $$\sqrt[5]{\frac{0.0075433^2 \times 78.343 \times 8172.4^{\frac{1}{3}} \times 0.00052}{64285.^{\frac{1}{2}} \times 154.27^4 \times 0.001 \times 586.79^{\frac{1}{3}}}}.$$

80. $$\sqrt[5]{\frac{15.832^3 \times 5793.6^{\frac{1}{3}} \times 0.78426}{0.000327^{\frac{1}{3}} \times 768.94^2 \times 3015.3 \times 0.007^{\frac{1}{3}}}}.$$

81. $$\sqrt[5]{\frac{7.1895 \times 4764.2^2 \times 0.00326^5}{0.00048953 \times 457^3 \times 5764.4^2}}.$$

82. $$\sqrt[5]{\frac{3.1416 \times 4771.21 \times 2.7183^{\frac{1}{3}}}{30.103^4 \times 0.4343^{\frac{1}{2}} \times 69.897^4}}.$$

83. $$\sqrt[7]{\frac{0.03271^2 \times 53.429 \times 0.77542^3}{32.769 \times 0.000371^4}}.$$

84. $$\sqrt[3]{\frac{732.056^2 \times 0.0003572^4 \times 89793}{42.2798^3 \times 3.4574 \times 0.0026518^5}}.$$

85. $$\sqrt[3]{\frac{7932 \times 0.00657 \times 0.80464}{0.03274 \times 0.6428}}.$$

86. $$\sqrt[3]{\frac{7.1206 \times \sqrt{0.13274} \times 0.057389}{\sqrt{0.43468} \times 17.385 \times \sqrt{0.0096372}}}.$$

87. $$\left\{\frac{3.075526^2 \times 5771.2^{\frac{1}{3}} \times 0.0036984^{\frac{1}{3}} \times 7.74}{72258 \times 327.93^3 \times 86.97^5}\right\}^{\frac{2}{3}}.$$

318. Since any positive number other than 1 may be taken as the base of a system of logarithms, the following general proofs to the base a should be noticed.

I. *The logarithm of the product of two or more numbers is equal to the sum of the logarithms of the numbers.*

For, let m and n be two numbers, and x and y their logarithms.

Then, by the definition of a logarithm, $m = a^x$, and $n = a^y$.

Hence, $$m \times n = a^x \times a^y = a^{x+y}.$$

$$\therefore \log (m \times n) = x + y,$$
$$= \log m + \log n.$$

In like manner, the proposition may be extended to any number of factors.

II. *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

For, let m and n be two numbers, and x and y their logarithms.

Then $$m = a^x, \text{ and } n = a^y.$$

Hence, $$m \div n = a^x \div a^y = a^{x-y}.$$

$$\therefore \log (m \div n) = x - y,$$
$$= \log m - \log n.$$

From this it follows that $\log \frac{1}{m} = \log 1 - \log m.$

But, since $$\log 1 = 0, \log \frac{1}{m} = -\log m.$$

III. *The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

For, let x be the logarithm of m.

Then $$m = a^x,$$

and $$m^p = (a^x)^p = a^{px}.$$

$$\therefore \log m^p = px,$$
$$= p \log m.$$

IV. *The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.*

For, let x be the logarithm of m.

Then $$m = a^x,$$

and $$m^{\frac{1}{r}} = (a^x)^{\frac{1}{r}} = a^{\frac{x}{r}}.$$

$$\therefore \log m^{\frac{1}{r}} = \frac{x}{r} = \frac{\log m}{r}.$$

319. An **exponential equation**, that is, an equation in which the exponent is the unknown quantity, is easily solved by logarithms.

For, let $$a^x = m.$$

Then $$\log a^x = \log m,$$

$$\therefore x \log a = \log m,$$

$$\therefore x = \frac{\log m}{\log a}.$$

Ex. Find the value of x in $81^x = 10$.

$$81^x = 10,$$

$$x = \frac{\log 10}{\log 81},$$

$$\therefore \log x = \log \log 10 + \text{colog} \log 81,$$

$$= 0 + 9.7193 - 10,$$

$$\therefore x = 0.524.$$

320. Logarithms of numbers to any base a may be converted into logarithms to any other base b by dividing the computed logarithms by the logarithm of b to the base a.

For, let $\log m = y$ to the base b,

and $\log b = x$ to the base a.

Then $$m = b^y, \text{ and } b = a^x,$$

$$\therefore m = (a^x)^y = a^{xy}.$$

$$\therefore \log m \text{ (to base } a) = xy = \log b \text{ (to base } a) \times \log m \text{ (to base } b),$$

$$\therefore \log m \text{ (to base } b) = \frac{\log m \text{ (to base } a)}{\log b \text{ (to base } a)}.$$

This is usually written, $\log_b m = \dfrac{\log_a m}{\log_a b}.$

CHAPTER XX.

RATIO, PROPORTION, AND VARIATION.

321. The *relative magnitude* of two numbers is called their **ratio,** and is expressed by the fraction which the first is of the second.

Thus, the ratio of 6 to 3 is indicated by the fraction $\frac{6}{3}$, which is sometimes written $6:3$.

322. The first term of a ratio is called the **antecedent,** and the second term the **consequent.** When the antecedent is *equal* to the consequent, the ratio is called a *ratio of equality;* when the antecedent is *greater* than the consequent, the ratio is called a *ratio of greater inequality;* when *less*, a *ratio of less inequality.*

323. When the antecedent and consequent are interchanged, the resulting ratio is called the *inverse* of the given ratio.

Thus, the ratio $3:6$ is the *inverse* of the ratio $6:3$.

324. The ratio of two *quantities* that can be expressed in *integers* in terms of a *common unit* is equal to the ratio of the two *numbers* by which they are expressed.

Thus, the ratio of \$9 to \$11 is equal to the ratio of $9:11$; and the ratio of a line $2\frac{2}{3}$ inches long to a line $3\frac{3}{4}$ inches long, when both are expressed in terms of a unit $\frac{1}{12}$ of an inch long, is equal to the ratio of 32 to 45.

325. Two quantities *different in kind* can have no ratio, for then one cannot be a fraction of the other.

326. Two quantities that can be expressed in integers in terms of a common unit are said to be *commensurable*. The common unit is called a *common measure*, and each quantity is called a *multiple* of this common measure.

Thus, a common measure of $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet is $\frac{1}{6}$ of a foot, which is contained 15 times in $2\frac{1}{2}$ feet, and 22 times in $3\frac{2}{3}$ feet. Hence, $2\frac{1}{2}$ feet and $3\frac{2}{3}$ feet are multiples of $\frac{1}{6}$ of a foot, $2\frac{1}{2}$ feet being obtained by taking $\frac{1}{6}$ of a foot 15 times, and $3\frac{2}{3}$ by taking $\frac{1}{6}$ of a foot 22 times.

327. When two quantities are *incommensurable*, that is, have no common unit in terms of which *both* quantities can be expressed in *integers*, it is impossible to find a fraction that will indicate the exact value of the ratio of the given quantities. It is possible, however, by taking the unit sufficiently small, to find a fraction that shall differ from the true value of the ratio by as little as we please.

Thus, if a and b denote the diagonal and side of a square,

$$\frac{a}{b} = \sqrt{2}.$$

Now $\sqrt{2} = 1.41421356.....$, a value greater than 1.414213, but less than 1.414214.

If, then, a *millionth part of b* be taken as the unit, the value of the ratio $\frac{a}{b}$ lies between $\frac{1414213}{1000000}$ and $\frac{1414214}{1000000}$, and therefore differs from either of these fractions by less than $\frac{1}{1000000}$.

By carrying the decimal farther, a fraction may be found that will differ from the true value of the ratio by less than a *billionth, trillionth, or any other assigned value whatever.*

328. Expressed generally, when a and b are incommensurable, and b is divided into any integral number (n) of equal parts, if one of these parts be contained in a more than m times, but less than $m+1$ times, then

$$\frac{a}{b} > \frac{m}{n}, \text{ but } < \frac{m+1}{n};$$

that is, the value of $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$.

The error, therefore, in taking either of these values for $\frac{a}{b}$ is $< \frac{1}{n}$. But by increasing n indefinitely, $\frac{1}{n}$ can be made to decrease indefinitely, and to become less than any assigned value, however small, though it cannot be made absolutely equal to zero.

329. The ratio between two incommensurable quantities is called an **incommensurable ratio.**

330. As the treatment of Proportion in Algebra depends upon the assumption that it is possible to find fractions which will *represent* the ratios, and as it appears that no fraction can be found to represent the exact value of an incommensurable ratio, it is necessary to show that *two incommensurable ratios are equal if their true values always lie between the same limits, however little these limits differ from each other.*

Let $a:b$ and $c:d$ be two incommensurable ratios.

Suppose the true values of the ratios $a:b$ and $c:d$ lie between $\frac{m}{n}$ and $\frac{m+1}{n}$. Then the *difference* between the true values of these ratios is *less* than $\frac{1}{n}$, however small the value of $\frac{1}{n}$ may be. § 328.

But since $\frac{1}{n}$ can be made to approach zero at pleasure, $\frac{1}{n}$ can be made *less* than *any assumed difference* between the ratios.

Therefore, to assume any difference between the ratios is to assume it possible to find a quantity that for the same value of $\frac{1}{n}$ shall be both *greater* and *less* than $\frac{1}{n}$; which is a manifest absurdity.

Hence, $a:b = c:d$.

331. It will be well to notice that the word **limit means** a fixed value from which another and variable value may be made to differ by as little as we please; it being impossible, however, for the difference between the variable value and the limit to become absolutely zero.

332. *A ratio will not be altered if both its terms be multiplied by the same number.*

For the ratio $a:b$ is represented by $\frac{a}{b}$, the ratio $ma:mb$ is represented by $\frac{ma}{mb}$; and since $\frac{ma}{mb}=\frac{a}{b}$, $\therefore ma:mb=a:b$.

333. *A ratio will be altered if different multipliers of its terms be taken; and will be increased or diminished according as the multiplier of the antecedent is greater or less than that of the consequent.*

$$\begin{array}{rl}
\text{For,} & ma:nb \text{ will be} > \text{or} < a:b \\
\text{according as} & \frac{ma}{nb} \text{ is} > \text{or} < \frac{a}{b}\left(=\frac{na}{nb}\right), \\
\text{as} & ma \text{ is} > \text{or} < na, \\
\text{as} & m \text{ is} > \text{or} < n.
\end{array}$$

334. *A ratio of greater inequality will be diminished, and a ratio of less inequality increased by adding the same number to both its terms.*

$$\begin{array}{rl}
\text{For,} & a+x:b+x \text{ is} > \text{or} < a:b \\
\text{according as} & \frac{a+x}{b+x} \text{ is} > \text{or} < \frac{a}{b}, \\
\text{as} & ab+bx \text{ is} > \text{or} < ab+ax, \\
\text{as} & bx \text{ is} > \text{or} < ax, \\
\text{as} & b \text{ is} > \text{or} < a.
\end{array}$$

335. *A ratio of greater inequality will be increased, and a ratio of less inequality diminished, by subtracting the same number from both its terms.*

$$\begin{array}{rl}
 & \text{For, } a-x:b-x \text{ will be} > \text{or} < a:b \\
\text{according as} & \frac{a-x}{b-x} \text{ is} > \text{or} < \frac{a}{b}, \\
\text{as} & ab-bx \text{ is} > \text{or} < ab-ax, \\
\text{as} & ax \text{ is} > \text{or} < bx, \\
\text{as} & a \text{ is} > \text{or} < b.
\end{array}$$

336. Ratios are *compounded* by taking the product of the fractions that represent them.

Thus, the ratio compounded of $a:b$ and $c:d$ is found by taking the product of $\frac{a}{b}$ and $\frac{c}{d} = \frac{ac}{bd}$.

The ratio compounded of $a:b$ and $a:b$ is the *duplicate* ratio $a^2:b^2$, and the ratio compounded of $a:b$, $a:b$, and $a:b$ is the *triplicate* ratio $a^3:b^3$.

337. Ratios are *compared* by comparing the fractions that represent them.

Thus, $a:b$ is $>$ or $<$ $c:d$

according as $\frac{a}{b}$ is $>$ or $<$ $\frac{c}{d}$,

as $\frac{ad}{bd}$ is $>$ or $<$ $\frac{bc}{bd}$,

as ad is $>$ or $<$ bc.

Exercise CXIII.

1. Write down the ratio compounded of $3:5$ and $8:7$. Which of these ratios is increased, and which is diminished by the composition?

2. Compound the duplicate ratio of $4:15$ with the triplicate of $5:2$.

3. Show that a duplicate ratio is greater or less than its simple ratio according as it is a ratio of greater or less inequality.

4. Arrange in order of magnitude the ratios $3:4$; $23:25$; $10:11$; and $15:16$.

5. Arrange in order of magnitude
$$a+b:a-b \text{ and } a^2+b^2:a^2-b^2, \text{ if } a>b.$$

Find the ratios compounded of:

6. $3:5$; $10:21$; $14:15$.

7. $7:9$; $102:105$; $15:17$.

8. $\dfrac{a^2+ax+x^2}{a^3-a^2x+ax^2-x^3}$ and $\dfrac{a^2-ax+x^2}{a+x}$.

9. $\dfrac{x^2-9x+20}{x^2-6x}$ and $\dfrac{x^2-13x+42}{x^2-5x}$.

10. $a+b:a-b$; $a^2+b^2:(a+b)^2$; $(a^2-b^2)^2:a^4-b^4$.

11. Two numbers are in the ratio $2:3$, and if 9 be added to each, they are in the ratio $3:4$. Find the numbers.

(Let $2x$ and $3x$ represent the numbers).

12. Show that the ratio $a:b$ is the duplicate of the ratio $a+c:b+c$, if $c^2=ab$.

13. Find two numbers in the ratio $3:4$, of which the sum is to the sum of their squares in the ratio of 7 to 50.

14. If five gold coins and four silver ones be worth as much as three gold coins and twelve silver ones, find the ratio of the value of a gold coin to that of a silver one.

15. If eight gold and nine silver coins be worth as mucn as six gold and nineteen silver coins, find the ratio of the value of a silver coin to that of a gold one.

16. There are two roads from A to B, one of them 14 miles longer than the other; and two roads from B to C, one of them 8 miles longer than the other. The distance from A to B is to the distance from B to C, by the shorter roads, as 1 to 2; by the longer roads, as 2 to 3. Find the distances.

17. What must be added to each of the terms of the ratio $m:n$, that it may become equal to the ratio $p:q$?

18. A rectangular field contains 5270 acres, and its length is to its breadth in the ratio of $31:17$. Find its dimensions.

PROPORTION.

338. An equation consisting of two equal ratios is called a **proportion**; and the terms of the ratios are called **proportionals.**

339. The algebraic test of a proportion is that the two fractions which represent the ratios shall be equal.

Thus, the ratio $a:b$ will be equal to the ratio $c:d$ if $\frac{a}{b}=\frac{c}{d}$; and the four *numbers* a, b, c, d are called proportionals, or are said to be in proportion.

340. If the ratios $a:b$ and $c:d$ form a proportion, the proportion is written

$$a:b=c:d$$

(read the ratio of a to b is equal to the ratio of c to d)

or

$$a:b::c:d$$

(read a is to b in the same ratio as c is to d).

The first and last terms, a and d, are called the **extremes.**

The two middle terms, b and c, are called the **means.**

341. *When four numbers are in proportion, the product of the extremes is equal to the product of the means.*

For, if $\quad a:b::c:d,$

then $\quad \frac{a}{b}=\frac{c}{d}.$

By multiplying by bd, $\quad ad=bc.$

342. *If the product of two numbers be equal to the product of two others, either two may be made the extremes of a proportion and the other two the means.*

For, if $\quad ad=bc,$

by dividing by bd, $\quad \frac{ad}{bd}=\frac{bc}{bd},$

or $\quad \frac{a}{b}=\frac{c}{d}.$

$$\therefore a:b::c:d.$$

343. The equation $ad = bc$ gives

$$a = \frac{bc}{d}; \quad b = \frac{ad}{c};$$

so that an extreme may be found by dividing the product of the means by the other extreme; and a mean may be found by dividing the product of the extremes by the other mean.

344. If four quantities, a, b, c, d, be in proportion, they will be in proportion by:

I. **Inversion.**

That is, b will be to a as d is to c.

For, if $a : b :: c : d$,

then $\frac{a}{b} = \frac{c}{d}$,

and $1 + \frac{a}{b} = 1 + \frac{c}{d}$;

or $\frac{b}{a} = \frac{d}{c}$,

$\therefore b : a :: d : c$.

345. II. **Composition.**

That is, $a + b$ will be to b as $c + d$ is to d.

For, if $a : b :: c : d$,

then $\frac{a}{b} = \frac{c}{d}$,

and $\frac{a}{b} + 1 = \frac{c}{d} + 1$,

or $\frac{a+b}{b} = \frac{c+d}{d}$,

$\therefore a + b : b :: c + d : d$.

346. III. **Division.**

That is, $a - b$ will be to b as $c - d$ is to d.

For, if $a : b :: c : d$,

then $\frac{a}{b} = \frac{c}{d}$,

and $$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or $$\frac{a-b}{b} = \frac{c-d}{d},$$

$$\therefore a - b : b :: c - d : d.$$

347. IV. Composition and Division.

That is, $a + b$ will be to $a - b$ as $c + d$ is to $c - d$.

For, from II., $$\frac{a+b}{b} = \frac{c+d}{d},$$

and from III., $$\frac{a-b}{b} = \frac{c-d}{d}.$$

By dividing, $$\frac{a+b}{a-b} = \frac{c+d}{c-d},$$

$$\therefore a + b : a - b :: c + d : c - d.$$

348. When the four quantities a, b, c, d are all of the *same kind*, they will be in proportion by :

V. **Alternation.**

That is, a will be to c as b is to d.

For, if $$a : b :: c : d,$$

then $$\frac{a}{b} = \frac{c}{d}.$$

By multiplying by $\frac{b}{c}$, $$\frac{ab}{bc} = \frac{bc}{cd},$$

or $$\frac{a}{c} = \frac{b}{d},$$

$$\therefore a : c :: b : d.$$

349. From the proportion $a : c :: b : d$ may be obtained by :

VI. **Composition.** $a + c : c :: b + d : d.$

VII. **Division.** $a - c : c :: b - d : d.$

VIII. **Composition and Division.** $a + c : a - c :: b + d : b - d.$

350. *In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.*

For, if $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h},$$
r may be put for each of these ratios.

Then $$\frac{a}{b}=r,\ \frac{c}{d}=r,\ \frac{e}{f}=r,\ \frac{g}{h}=r,$$
$$\therefore a=br,\ c=dr,\ e=fr,\ g=hr.$$
$$\therefore a+c+e+g=(b+d+f+h)\,r.$$
$$\therefore \frac{a+c+e+g}{b+d+f+h}=r=\frac{a}{b}.$$
$$\therefore a+c+e+g : b+d+f+h :: a : b.$$

In like manner, it may be shown that
$$ma+nc+pe+qg : mb+nd+pf+qh :: a : b.$$

351. If a, b, c, d be in *continued* proportion, that is, if $a:b=b:c=c:d$, then will $a:c=a^2:b^2$ and $a:d=a^3:b^3$.

For, $$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}.$$

Hence, $$\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b},$$

or $$\frac{a}{c}=\frac{a^2}{b^2},$$
$$\therefore a:c=a^2:b^2.$$

So $$\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b},$$

or $$\frac{a}{d}=\frac{a^3}{b^3},$$
$$\therefore a:d=a^3:b^3.$$

352. If a, b, c be proportionals, so that $a:b::b:c$, then b is called a *mean* proportional between a and c, and c is called a *third* proportional to a and b.

If $a : b :: b : c$, then $b = \sqrt{ac}$.

For, if $a : b :: b : c$,

then $\frac{a}{b} = \frac{b}{c}$,

and $b^2 = ac$,

$\therefore b = \sqrt{ac}$.

353. *The products of the corresponding terms of two or more proportions are in proportion.*

For, if $a : b :: c : d$,

$e : f :: g : h$,

and $k : l :: m : n$,

then $\frac{a}{b} = \frac{c}{d}$, $\frac{e}{f} = \frac{g}{h}$, $\frac{k}{l} = \frac{m}{n}$.

Hence, by finding the product of the left members, and also of the right members of these equations,

$$\frac{aek}{bfl} = \frac{cgm}{dhn},$$

$$\therefore aek : bfl :: cgm : dhn.$$

354. *Like powers, or like roots, of the terms of a proportion are in proportion.*

For, if $a : b :: c : d$,

then $\frac{a}{b} = \frac{c}{d}$.

By raising both sides to the *nth* power,

$$\frac{a^n}{b^n} = \frac{c^n}{d^n},$$

$$\therefore a^n : b^n :: c^n : d^n.$$

By extracting the *nth* root,

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}},$$

$$\therefore a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}.$$

355. *If two quantities be increased or diminished by like parts of each, the results will be in the same ratio as the quantities themselves.*

For, $$\frac{a}{b}=\frac{\left(1\pm\frac{m}{n}\right)a}{\left(1\pm\frac{m}{n}\right)b},$$

that is, $$\frac{a}{b}=\frac{a\pm\frac{m}{n}a}{b\pm\frac{m}{n}b},$$

$$\therefore a:b::a\pm\frac{m}{n}a:b\pm\frac{m}{n}b.$$

356. The laws that have been established for ratios should be rememembered when ratios are expressed in their fractional form.

(1) Solve: $$\frac{x^2+x+1}{x^2-x-1}=\frac{x^2-x+2}{x^2+x-2}.$$

By § 347 $$\frac{2x^2}{2(x+1)}=\frac{2x^2}{-2(x-2)},$$
and this equation is satisfied, when $x=0$;
or, dividing by $\frac{2x^2}{2}$, $$\frac{1}{x+1}=\frac{1}{2-x},$$
$$\therefore x=\tfrac{1}{2}.$$

(2) If $a:b::c:d$, show that
$$a^2+ab:b^2-ab::c^2+cd:d^2-cd.$$

If $$\frac{a}{b}=\frac{c}{d},$$

then $$\frac{a+b}{a-b}=\frac{c+d}{c-d},$$ § 347.

and $$\frac{a}{-b}=\frac{c}{-d},$$

$$\therefore\ \frac{a}{-b}\times\frac{a+b}{a-b}=\frac{c}{-d}\times\frac{c+d}{c-d};$$ § 353.

that is, $$\frac{a^2+ab}{b^2-ab}=\frac{c^2+cd}{d^2-cd},$$

or $$a^2+ab:b^2-ab::c^2+cd:d^2-cd.$$

(3) When $a : b :: c : d$, and a is the *greatest term*, show that $a + d$ is greater than $b + c$.

Since $\frac{a}{b} = \frac{c}{d}$, and $a > c$,

$\therefore b > d.$

Also, since $\frac{a-b}{b} = \frac{c-d}{d}$, § 346.

and $b > d$,

$\therefore a - b > c - d.$

By adding, $b + d = b + d$,

$a + d > b + c.$

Exercise CXIV.

If $a : b :: c : d$, prove that:

1. $ma : nb :: mc : nd$.
2. $3a + b : b :: 3c + d : d$.
3. $a + 2b : b :: c + 2d : d$.
4. $a^3 : b^3 :: c^3 : d^3$.
5. $a : a + b :: c : c + d$.
6. $a : a - b :: c : c - d$.
7. $ma + nb : ma - nb :: mc + nd : mc - nd$.
8. $2a + 3b : 3a - 4b :: 2c + 3d : 3c - 4d$.
9. $ma^2 + nc^2 : mb^2 + nd^2 :: a^2 : b^2$.
10. $ma^2 + nab + pb^2 : mc^2 + ncd + pd^2 :: b^2 : d^2$.

If $a : b :: b : c$, prove that:

11. $a + b : b + c :: a : b$.
12. $a^2 + ab : b^2 + bc :: a : c$.
13. $a : c :: (a + b)^2 : (b + c)^2$.
14. When a, b, and c are proportionals, and a the greatest, show that $a + c > 2b$.
15. If $\frac{x-y}{l} = \frac{y-z}{m} = \frac{z-x}{n}$, and x, y, z be unequal, then $l + m + n = 0$.

16. Find x when $x+5 : 2x-3 :: 5x+1 : 3x-3$.

17. Find x when $x+a : 2x-b :: 3x+b : 4x-a$.

18. Find x when $\sqrt{x}+\sqrt{b} : \sqrt{x}-\sqrt{b} :: a : b$.

19. Find x and y when $x : 27 :: y : 9$, and $x : 27 :: 2 : x-y$.

20. Find x and y when $x+y+1 : x+y+2 :: 6 : 7$, and when $y+2x : y-2x :: 12x+6y-3 : 6y-12x-1$.

21. Find x when $x^2-4x+2 : x^2-2x-1 :: x^2-4x : x^2-2x-2$.

22. A railway passenger observes that a train passes him, moving in the opposite direction, in 2 seconds; but moving in the same direction with him, it passes him in 30 seconds. Compare the rates of the two trains.

23. A and B trade with different sums. A gains \$200 and B loses \$50, and now A's stock : B's :: 2 : ½. But, if A had gained \$100 and B lost \$85, their stocks would have been as 15 : 3½. Find the original stock of each.

24. A quantity of milk is increased by watering in the ratio 4 : 5, and then 3 gallons are sold; the remainder is mixed with 3 quarts of water, and is increased in the ratio 6 : 7. How many gallons of milk were there at first?

25. In a mile race between a bicycle and a tricycle their rates were as 5 : 4. The tricycle had half a minute start, but was beaten by 176 yards. Find the rates of each.

26. The time which an express-train takes to travel 180 miles is to that taken by an ordinary train as 9 : 14. The ordinary train loses as much time from stopping as it would take to travel 30 miles; the express-train loses only half as much time as the other by stopping, and travels 15 miles an hour faster. What are their respective rates?

27. A line is divided into two parts in the ratio 2 : 3, and into two parts in the ratio 3 : 4; the distance between the points of section is 2. Find the length of the line.

28. A railway consists of two sections; the annual expenditure on one is increased this year 5%, and on the other 4%, producing on the whole an increase of $4\frac{3}{10}$%. Compare the amount expended on the two sections last year, and also this year.

29. When a, b, c, d are proportional and unequal, show that no number x can be found such that $a+x$, $b+x$, $c+x$, $d+x$ shall be proportionals.

Variation.

357. Two quantities may be so related that, when one has its value changed, the other will, in consequence, have its value changed.

Thus, the *distance* travelled in a certain time will be *doubled* if the *rate* be *doubled*. The *time* required for doing a certain quantity of work will be *doubled* if only *half* the number of *workmen* be employed.

358. Whenever it becomes necessary to express the *general relations* of certain kinds of quantities to each other, without confining the inquiry to any *particular values* of these quantities, it will usually be sufficient to mention two of the terms of a proportion. In all such cases, however, four terms are always implied.

Thus, if it be said that the weight of water is proportional to its volume, or varies as its volume, the meaning is, that *one* gallon of water is to *any number* of gallons as the weight of *one* gallon is to the weight of the *given number* of gallons.

359. Quantities used in a *general* sense, as distance, time, weight, volume, to which particular values may be assigned, are denoted by capital letters, A, B, C, etc.; while *assigned values* of these quantities may be denoted by small letters, a, b, c, etc. The letters A, B, C will be understood to represent *any numerical values* that may be assigned to the quantities; and when two such letters occur in an expression they will be understood to represent *any corresponding* numerical values that may be assigned to the two quantities.

360. When two quantities A and B are so connected that their ratio is *constant*, that is, remains the same for all corresponding values of A and B, the one is said to *vary as* the other; and this relation is expressed by $A \propto B$ (read *A varies as B*).

Thus, the area of a triangle with a given base varies as its altitude; for, if the altitude be changed, the area will be changed in the same ratio.

If this constant ratio be denoted by m, then $\frac{A}{B} = m$, or $A = mB$.

From this equation m may be found when two corresponding values of A and B are known.

361. When two quantities are so connected that if one be changed in any ratio, the other will be changed in the *inverse* ratio, the one is said to *vary inversely as* the other.

Thus, the time required to do a certain amount of work varies inversely as the number of workmen employed; for, if the number of workmen be doubled, halved, or changed in any ratio, the time required will be halved, doubled, or changed in the inverse ratio.

362. If A vary inversely as B, two values of A have to each other the inverse ratio of the two corresponding values of B; or $a : a' :: b' : b$; that is, $ab = a'b'$.

Hence, the product AB is constant, and may be denoted by m. That is, $AB = m$.

If any two corresponding values of A and B be known, the constant m may be found.

The equation $AB = m$ may be written $A = \frac{m}{B}$, and as m is constant, A is said to vary as the *reciprocal of* B, or $A \propto \frac{1}{B}$.

363. The two equations,

$$A = mB \text{ (for direct variation),}$$
$$A = \frac{m}{B} \text{ (for inverse variation),}$$

furnish the simplest method of treating Variation.

If $A = mBC$, A is said to vary *jointly* as B and C.

If $A = \frac{mB}{C}$, A is said to vary *directly* as B and *inversely* as C.

364. The following results are to be observed:

I. If $A \propto B$ and $B \propto C$, then $A \propto C$.

For $A = mB$, where m is constant,
and $B = nC$, where n is constant.
$\therefore A = mnC$.
$\therefore A \propto C$, since mn is constant.

In like manner, if $A \propto B$ and $B \propto \frac{1}{C}$, then $A \propto \frac{1}{C}$.

II. If $A \propto C$ and $B \propto C$, then $A \pm B \propto C$, and $\sqrt{AB} \propto C$.

For $A = mC$, where m is constant,
and $B = nC$, where n is constant.
$\therefore A \pm B = (m \pm n) C$.
$\therefore A \pm B \propto C$, since $m \pm n$ is constant.
Also, $\sqrt{AB} = \sqrt{mC \times nC} = \sqrt{mnC^2} = C\sqrt{mn}$.
$\therefore \sqrt{AB} \propto C$, since $\sqrt{mn}$ is constant.

III. If $A \propto B$ and $C \propto D$, then $AC \propto BD$.

For $\quad A = mB$, where m is constant,

$C = nD$, where n is constant.

$\therefore AC = mnBD$.

$\therefore AC \propto BD$, since mn is constant.

IV. If $A \propto B$ then $A^n \propto B^n$.

For $\quad A = mB$, where m is constant.

$\therefore A^n = m^nB^n$.

$\therefore A^n \propto B^n$, since m^n is constant.

V. If $A \propto B$ when C is unchanged, and $A \propto C$ when B is unchanged, then $A \propto BC$ when both B and C change.

For $\quad A = mB$, when B varies and C is constant.
Here, m is *constant* and cannot contain the *variable* B,
$\therefore A$ *must* contain B, but no other power of B.
Again, $\quad A = nC$, when C varies and B is constant.
Here, n is *constant* and cannot contain the *variable* C,
$\therefore A$ must contain C, but no other power of C.
Hence, A contains both B and C, but no other powers of B and C, and therefore,

$$\frac{A}{BC} = p, \text{ or } A = pBC, \text{ where } p \text{ is constant.}$$

$$\therefore A \propto BC, \text{ since } p \text{ is constant.}$$

In like manner, it may be shown that if A vary as each of any number of quantities B, C, D, etc., when the rest are unchanged, then when they all change, $A \propto BCD$, etc.

Thus, the area of a rectangle varies as the base when the altitude is constant, and as the altitude when the base is constant, but as the product of the base and altitude when both vary.

The volume of a rectangular solid varies as the length when the width and thickness remain constant; as the width when the length and thickness remain constant; as the thickness when the length and width remain constant; but as the product of length, breadth, and thickness when all three vary.

(1) If A vary inversely as B, and when $A=2$ the corresponding value of B is 36, find the corresponding value of B when $A=9$.

$$\text{Here} \qquad A=\frac{m}{B},$$
$$\text{or} \qquad m=AB,$$
$$\therefore m=2\times 36=72.$$

And if 9 and 72 be substituted for A and m respectively in

$$A=\frac{m}{B},$$
$$\text{the result is} \qquad 9=\frac{72}{B},$$
$$\therefore 9B=72.$$
$$\therefore B=8. \quad \textit{Ans.}$$

(2) The weight of a sphere of given material varies as its volume, and its volume varies as the cube of its diameter. If a sphere 4 inches in diameter weigh 20 pounds, find the weight of a sphere 5 inches in diameter.

Let W represent the weight,
V represent the volume,
D represent the diameter.

Then $W \propto V$ and $V \propto D^3$,

$$\therefore W \propto D^3.$$

Put $W=mD^3$,

then, since 20 and 4 are corresponding volumes of W and D,

$$20=m\times 64,$$
$$\therefore m=\tfrac{20}{64}=\tfrac{5}{16}.$$
$$\therefore W=\tfrac{5}{16}D^3.$$
$$\therefore \text{when } D=5, W=\tfrac{5}{16} \text{ of } 125=39\tfrac{1}{16}.$$

Exercise CXV.

1. If $A \propto B$, and $A=4$ when $B=5$, find A when $B=12$.
2. If $A \propto B$, and when $B=\frac{1}{2}$, $A=\frac{1}{3}$, find A when $B=\frac{1}{3}$.
3. If A vary jointly as B and C, and 3, 4, 5 be simultaneous values of A, B, C, find A when $B=C=10$.

4. If $A \propto \frac{1}{B}$, and when $A = 10$, $B = 2$, find the value of B when $A = 4$.

5. If $A \propto \frac{B}{C}$, and when $A = 6$, $B = 4$, and $C = 3$, find the value of A when $B = 5$ and $C = 7$.

6. If the square of X vary as the cube of Y, and $X = 3$ when $Y = 4$, find the equation between X and Y.

7. If the square of X vary inversely as the cube of Y, and $X = 2$ when $Y = 3$, find the equation between X and Y.

8. If Z vary as X directly and Y inversely, and if when $Z = 2$, $X = 3$, and $Y = 4$, find the value of Z when $X = 15$ and $Y = 8$.

9. If $A \propto B + c$ where c is constant, and if $A = 2$ when $B = 1$, and if $A = 5$ when $B = 2$, find A when $B = 3$.

10. The velocity acquired by a stone falling from rest varies as the time of falling; and the distance fallen varies as the square of the time. If it be found that in 3 seconds a stone has fallen 145 feet, and acquired a velocity of $96\frac{2}{3}$ feet per second, find the velocity and distance at the end of 5 seconds.

11. If a heavier weight draw up a lighter one by means of a string passing over a fixed wheel, the space described in a given time will vary directly as the difference between the weights, and inversely as their sum. If 9 ounces draw 7 ounces through 8 feet in 2 seconds, how high will 12 ounces draw 9 ounces in the same time?

12. The space will vary also as the square of the time. Find the space in Example 11, if the time in the latter case be 3 seconds.

13. Equal volumes of iron and copper are found to weigh 77 and 89 ounces respectively. Find the weight of $10\frac{1}{2}$ feet of round copper rod when 9 inches of iron rod of the same diameter weigh $31\frac{9}{10}$ ounces.

14. The square of the time of a planet's revolution varies as the cube of its distance from the sun. The distances of the Earth and Mercury from the sun being 91 and 35 millions of miles, find in days the time of Mercury's revolution.

15. A spherical iron shell 1 foot in diameter weighs $\frac{91}{216}$ of what it would weigh if solid. Find the thickness of the metal, it being known that the volume of a sphere varies as the cube of its diameter.

16. The volume of a sphere varies as the cube of its diameter. Compare the volume of a sphere 6 inches in diameter with the sum of the volumes of three spheres whose diameters are 3, 4, 5 inches respectively.

17. Two circular gold plates, each an inch thick, the diameters of which are 6 inches and 8 inches respectively, are melted and formed into a single circular plate 1 inch thick. Find its diameter, having given that the area of a circle varies as the square of its diameter.

18. The volume of a pyramid varies jointly as the area of its base and its altitude. A pyramid, the base of which is 9 feet square, and the height of which is 10 feet, is found to contain 10 cubic yards. What must be the height of a pyramid upon a base 3 feet square, in order that it may contain 2 cubic yards?

CHAPTER XXI.

SERIES.

365. A succession of numbers which proceed according to some fixed law is called a **series**; and the successive numbers are called the **terms** of the series.

Thus, by executing the indicated division of $\frac{1}{1-x}$, the series $1+x+x^2+x^3+\dots\dots$ is obtained, a series that has an *unlimited* number of terms.

366. A series that is continued *indefinitely* is called an **infinite series**; and a series that comes to an end at some particular term is called a **finite series.**

367. When x is <1, the more terms we take of the infinite series $1+x+x^2+x^3+\dots\dots$, obtained by dividing 1 by $1-x$, the more nearly does their sum *approach* to the value of $\frac{1}{1-x}$.

Thus, if $x=\frac{1}{3}$, then $\frac{1}{1-x}=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$, and the series becomes $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots\dots$, a sum which cannot become equal to $\frac{3}{2}$ however great the number of terms taken, but which may be made to differ from $\frac{3}{2}$ by as little as we please by increasing indefinitely the number of terms.

368. But when x is >1, the more terms we take of the series $1+x+x^2+x^3+\dots\dots$ the more does the sum of the series *diverge* from the value of $\frac{1}{1-x}$.

Thus, if $x=3$, then $\frac{1}{1-x}=\frac{1}{1-3}=-\frac{1}{2}$, and the series becomes $1+3+9+27+\dots\dots$, a sum which *diverges* more and more from $-\frac{1}{2}$,

the more terms we take, and which may be made to increase indefinitely by increasing indefinitely the number of terms taken.

369. A series whose sum as the number of its terms is indefinitely increased approaches some *fixed finite value as a limit* is called a **converging series**; and a series whose sum increases indefinitely as the number of its terms is increased, is called a **diverging series.**

370. When $x=1$, the division of 1 by $1-x$, that is, of 1 by 0, has no meaning, according to the *definition of division;* and any attempt to divide by a divisor that is equal to zero leads to absurd results.

Thus,	$8+4=8+4$;	
by transposing,	$8-8=4-4$;	
or, dividing by $4-4$,	$2=1$;	a manifest absurdity.

371. When $x=1$ *very nearly*, then the value of $\dfrac{1}{1-x}$ will be *very great*, and the sum of the series $1+x+x^2+x^3+\ldots\ldots$ will become greater and greater the more terms we take. Hence, by making the denominator $1-x$ approach indefinitely to zero, the value of the fraction $\dfrac{1}{1-x}$ may be made to increase at pleasure.

372. If the symbol $\circ$ be used to denote a quantity that is less than any assignable quantity, and that may be considered to decrease without limit, not, however, becoming 0, and the symbol $\underline{\infty}$ be used to denote a quantity that is greater than any assignable quantity, and that may be considered to increase without limit, not, however, becoming ∞, then

$$\frac{1}{\circ}=\underline{\infty}.$$

In the same sense $\dfrac{a}{\circ}=\underline{\infty}$, where a represents any value that may be assigned.

373. If x in the fraction $\frac{1-x^5}{1-x}$ be equal to 1, the numerator and denominator will each become 0, and the fraction will assume the form $\frac{0}{0}$.

374. If, however, x in this fraction approach to 1 as its limit, then the denominator $1-x$, inasmuch as it has *some* value, even though less than any assignable value, may be used as a divisor, and the result is $1+x+x^2+x^3+x^4$. Hence, it is evident that though both terms of the fraction become smaller and smaller as $1-x$ approaches to 0, still the numerator becomes more and more nearly *five times* the denominator.

It may be remarked that when the symbol $\frac{0}{0}$ is obtained for the value of the unknown quantity in a problem, the meaning is that the problem has no *definite* solution, but that its conditions are satisfied if any value whatever be taken for the required quantity; and if the symbol $\frac{a}{0}$, in which a denotes any assigned value, be obtained for the value of the unknown quantity, the meaning is that the conditions of the problem are impossible.

375. The number of different series is unlimited, but the only kinds of series that will be considered at this stage of the work are Arithmetical, Geometrical, and Harmonical Series.

Arithmetical Series.

376. A series in which the difference between any two adjacent terms is equal to the difference between any other two adjacent terms, is called an **Arithmetical Series** or an **Arithmetical Progression.**

377. The general representative of such a series will be

$$a,\ a+d,\ a+2d,\ a+3d.....,$$

in which a is the first term and d the common difference;

and the series will be *increasing* or *decreasing* according as d is positive or negative.

378. Since each succeeding term of the series is obtained by adding d to the preceding term, the coefficient of d will always be 1 less than the number of the term, so that the

$$nth\ \text{term} = a + (n - 1)\, d.$$

If the *nth* term be denoted by l, this equation becomes

$$l = a + (n - 1)\, d. \tag{1}$$

379. The *arithmetical mean* between two numbers is the number which stands between them, and makes with them an arithmetical series.

380. If a and b denote two numbers, and A their arithmetical mean, then, by the definition of an arithmetical series,

$$A - a = b - A,$$

$$\therefore A = \frac{a + b}{2}. \tag{2}$$

381. Sometimes it is required to insert several arithmetical means between two numbers.

If $m =$ the number of means, then $m + 2 = n$, the whole number of terms; and if $m + 2$ be substituted for n in the equation

$$l = a + (n - 1)\, d,$$

the result is

$$l = a + (m + 1)\, d.$$

By transposing a,

$$l - a = (m + 1)\, d,$$

$$\therefore \frac{l - a}{m + 1} = d. \tag{3}$$

Thus, if it be required to insert six means between 3 and 17, the value of d is found to be $\frac{17 - 3}{6 + 1} = 2$; and the series will be 3, 5, 7, 9, 11, 13, 15, 17.

382. If l denote the last term, a the first term, n the number of terms, d the common difference, and s the sum of the terms, it is evident that

$$\begin{aligned} s &= a + (a+d) + (a+2d) + \cdots + (l-d) + l, \text{ or} \\ s &= l + (l-d) + (l-2d) + \cdots + (a+d) + a \\ \therefore 2s &= (a+l) + (a+l) + (a+l) + \cdots + (a+l) + (a+l) \\ &= n(a+l) \end{aligned}$$

$$\therefore s = \frac{n}{2}(a+l). \qquad (4)$$

383. From the two equations,

$$l = a + (n-1)d, \qquad (1)$$

$$s = \frac{n}{2}(a+l), \qquad (2)$$

any one of the quantities a, d, l, n, s may be found when *three* are given.

Ex. Find n when d, l, s are given.

From (1), $a = l - (n-1)d$.

From (2), $a = \frac{2s - ln}{n}$.

Therefore, $l - (n-1)d = \frac{2s - ln}{n}$,

$$\therefore ln - dn^2 + dn = 2s - ln,$$

$$\therefore dn^2 - (2l + d)n = -2s,$$

$$\therefore 4d^2n^2 - (\quad) + (2l+d)^2 = (2l+d)^2 - 8ds,$$

$$\therefore 2dn - (2l+d) = \pm\sqrt{(2l+d)^2 - 8ds},$$

$$\therefore n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}.$$

NOTE. The table on the following page contains the results of the general solution of all possible problems in arithmetical series. The student is advised to work these out, both for the results obtained and for the practice gained in solving literal equations in which the unknown quantities are represented by other letters than x, y, z.

No.	Given.	Required.	Results.
1	$a\ d\ n$	l	$l = a + (n-1)d.$
2	$a\ d\ s$		$l = -\frac{1}{2}d \pm \sqrt{[2ds + (a - \frac{1}{2}d)^2]}.$
3	$a\ n\ s$		$l = \frac{2s}{n} - a.$
4	$d\ n\ s$		$l = \frac{s}{n} + \frac{(n-1)d}{2}.$
5	$a\ d\ n$	s	$s = \frac{1}{2}n[2a + (n-1)d].$
6	$a\ d\ l$		$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2d}.$
7	$a\ n\ l$		$s = (l+a)\frac{n}{2}.$
8	$d\ n\ l$		$s = \frac{1}{2}n[2l - (n-1)d].$
9	$d\ n\ l$	a	$a = l - (n-1)d.$
10	$d\ n\ s$		$a = \frac{s}{n} - \frac{(n-1)d}{2}.$
11	$d\ l\ s$		$a = \frac{1}{2}d \pm \sqrt{(l + \frac{1}{2}d)^2 - 2ds}.$
12	$n\ l\ s$		$a = \frac{2s}{n} - l.$
13	$a\ n\ l$	d	$d = \frac{l-a}{n-1}.$
14	$a\ n\ s$		$d = \frac{2(s-an)}{n(n-1)}.$
15	$a\ l\ s$		$d = \frac{l^2 - a^2}{2s - l - a}.$
16	$n\ l\ s$		$d = \frac{2(nl-s)}{n(n-1)}.$
17	$a\ d\ l$	n	$n = \frac{l-a}{d} + 1.$
18	$a\ d\ s$		$n = \frac{d - 2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}.$
19	$a\ l\ s$		$n = \frac{2s}{l+a}.$
20	$d\ l\ s$		$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}.$

Exercise CXVI.

1. Find the thirteenth term of 5, 9, 13.....
   ninth term of $-3, -1, 1$.....
   tenth term of $-2, -5, -8$.....
   eighth term of $a, a+3b, a+6b$.....
   fifteenth term of $1, \frac{6}{7}, \frac{5}{7}$.....
   thirteenth term of $-48, -44, -40$.....

2. The first term of an arithmetical series is 3, the thirteenth term is 55. Find the common difference.

3. Find the arithmetical mean between: (*a*.) 3 and 12; (*b*.) -5 and 17; (*c*.) a^2+ab-b^2 and a^2-ab+b^2.

4. Insert three arithmetical means between 1 and 19; and four means between -4 and 17.

5. The first term of a series is 2, and the common difference $\frac{1}{3}$. What term will be 10?

6. The seventh term of a series, whose common difference is 3, is 11. Find the first term.

7. Find the sum of

   $5+8+11+$..... to ten terms.
   $-4-1+2+$..... to seven terms.
   $a+4a+7a+$..... to n terms.
   $\frac{2}{3}+\frac{7}{15}+\frac{4}{15}+$..... to twenty-one terms.
   $1+2\frac{2}{3}+4\frac{1}{3}+$..... to twenty terms.

8. The sum of six numbers of an arithmetical series is 27, and the first term is 1. Determine the series.

9. How many terms of the series $-5-2+1+$..... must be taken so that their sum may be 63.

10. The first term is 12, and the sum of ten terms is 10. Find the last term.

11. The arithmetical mean between two numbers is 10, and the mean between the double of the first and the triple of the second is 27. Find the numbers.

12. Find the middle term of eleven terms whose sum is 66.

13. The first term of an arithmetical series is 2, the common difference is 7, and the last term 79. Find the number of terms.

14. The sum of fifteen terms of an arithmetical series is 600, and the common difference is 5. Find the first term.

15. Insert ten arithmetical means between -7 and 114.

16. The sum of three numbers in arithmetical progression is 15, and the sum of their squares is 83. Find the numbers.

Let $x-y$, x, $x+y$ represent the numbers.

17. Arithmetical means are inserted between 5 and 23, so that the sum of the first two is to the sum of the last two as 2 is to 5. How many means are inserted?

18. Find three numbers of an arithmetical series whose sum shall be 21, and the sum of the first and second shall be $\frac{3}{4}$ of the sum of the second and third.

19. Find three numbers whose common difference is 1, such that the product of the second and third exceeds that of the first and second by $\frac{1}{2}$.

20. How many terms of the series 1, 4, 7..... must be taken, in order that the sum of the first half may bear to the sum of the second half the ratio 10 : 31?

21. A travels uniformly 20 miles a day; B starts three days later, and travels 8 miles the first day, 12 the second, and so on, in arithmetical progression. In how many days will B overtake A?

22. A number consists of three digits which are in arithmetical progression; and this number divided by the sum of its digits is equal to 26; but if 198 be added to it, the digits in the units' and hundreds' places will be interchanged. Required the number.

23. The sum of the squares of the extremes of four numbers in arithmetical progression is 200, and the sum of the squares of the means is 136. What are the numbers?

24. Show that if any even number of terms of the series 1, 3, 5..... be taken, the sum of the first half is to the sum of the second half in the ratio 1 : 3.

25. A and B set out at the same time to meet each other from two places 343 miles apart. Their daily journeys are in arithmetical progression, A's increase being 2 miles each day, and B's decrease being 5 miles each day. On the day at the end of which they met, each travelled exactly 20 miles. Find the duration of the journey.

26. Suppose that a body falls through a space of $16\frac{1}{12}$ feet in the first second of its fall, and in each succeeding second $32\frac{1}{6}$ more than in the next preceding one. How far will a body fall in 20 seconds?

27. The sum of five numbers in arithmetical progression is 45, and the product of the first and fifth is $\frac{5}{8}$ of the the product of the second and fourth. Find the numbers.

28. If a full car descending an incline draw up an empty one at the rate of $1\frac{1}{2}$ feet the first second, $4\frac{1}{2}$ feet the next second, $7\frac{1}{2}$ feet the third, and so on, how long will it take to descend an incline 150 feet in length? What part of the distance will the car have descended in the first half of the time?

GEOMETRICAL SERIES.

384. A series is called a **Geometrical Series** or a **Geometrical Progression** when each succeeding term is obtained by multiplying the preceding term by a *constant multiplier.*

385. The general representative of such a series will be

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4 \ldots\ldots,$$

in which a is the first term and r the constant multiplier or ratio.

386. Since the exponent of r increases by 1 for every term, the exponent will always be 1 less than the number of the term; so that the

$$nth \text{ term} = ar^{n-1}.$$

387. If the nth term be denoted by l, this equation becomes

$$l = ar^{n-1}. \qquad (1)$$

388. The *geometrical mean* between two numbers is the number which stands between them, and makes with them a geometrical series.

389. If a and b denote two numbers, and G their geometrical mean, then, by definition of a geometrical series,

$$\frac{G}{a} = \frac{b}{G},$$

$$\therefore G = \sqrt{ab}. \qquad (2)$$

390. Sometimes it is required to insert several geometrical means between two numbers.

If m = the number of means, then $m+2=n$, the whole number of terms; and if $m+2$ be substituted for n in the equation

$$l=ar^{n-1},$$

the result is

$$l=ar^{m+1},$$

$$\therefore r^{m+1}=\frac{l}{a}. \qquad (3)$$

Thus, if it be required to insert three geometrical means between 3 and 48, the value of r is found to be

$$r^4=\frac{48}{3}=16.$$

$$\therefore r=2,$$

and the series will be $\quad 3, 6, 12, 24, 48.$

391. If l denote the last term, a the first term, n the number of terms, r the common ratio, and s the sum of the n terms, then

$$s=a+ar+ar^2+ar^3+\cdots\cdots+ar^{n-1}.$$

Multiply by r, $rs=ar+ar^2+ar^3+\cdots\cdots+ar^{n-1}+ar^n.$

Therefore, by subtracting the first equation from the second,

$$rs-s=ar^n-a,$$

or

$$(r-1)\,s=a\,(r^n-1),$$

$$\therefore s=\frac{a\,(r^n-1)}{r-1}. \qquad (4)$$

392. When r is <1, this formula will be more convenient if written

$$s=\frac{a\,(1-r^n)}{1-r}.$$

393. Since

$$l=ar^{n-1},$$

$$rl=ar^n,$$

and (4) may be written

$$s=\frac{rl-a}{r-1}.$$

In working out the following results, the student will make use of the two equations, $l=ar^{n-1}$ and $s=\frac{a\,(r^n-1)}{r-1}$.

No.	Given.	Required.	Results.
1	$a\ r\ n$	l	$l = ar^{n-1}$.
2	$a\ r\ s$		$l = \frac{a + (r-1)s}{r}$.
3	$a\ n\ s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$.
4	$r\ n\ s$		$l = \frac{(r-1)sr^{n-1}}{r^n - 1}$.
5	$a\ r\ n$	s	$s = \frac{a(r^n-1)}{r-1}$.
6	$a\ r\ l$		$s = \frac{rl - a}{r-1}$.
7	$a\ n\ l$		$s = \frac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}$.
8	$r\ n\ l$		$s = \frac{lr^n - l}{r^n - r^{n-1}}$.
9	$r\ n\ l$	a	$a = \frac{l}{r^{n-1}}$.
10	$r\ n\ s$		$a = \frac{(r-1)s}{r^n - 1}$.
11	$r\ l\ s$		$a = rl - (r-1)s$.
12	$n\ l\ s$		$a(s-a)^{n-1} - l(s-l)^{n-1} = 0$.
13	$a\ n\ l$	r	$r = \sqrt[n-1]{\frac{l}{a}}$.
14	$a\ n\ s$		$r^n - \frac{s}{a}r + \frac{s-a}{a} = 0$.
15	$a\ l\ s$		$r = \frac{s-a}{s-l}$.
16	$n\ l\ s$		$r^n - \frac{s}{s-l}r^{n-1} + \frac{l}{s-l} = 0$.
17	$a\ r\ l$	n	$n = \frac{\log l - \log a}{\log r} + 1$.
18	$a\ r\ s$		$n = \frac{\log[a + (r-1)s] - \log a}{\log r}$.
19	$a\ l\ s$		$n = \frac{\log l - \log a}{\log(s-a) - \log(s-l)} + 1$.
20	$r\ l\ s$		$n = \frac{\log l - \log[lr - (r-1)s]}{\log r} + 1$.

EXERCISE CXVII.

1. Find the seventh term of 2, 6, 18.....
 sixth term of 3, 6, 12.....
 ninth term of 6, 3, $1\frac{1}{2}$.....
 eighth term of 1, -2, 4.....
 twelfth term of x^3, x^4, x^5.....
 fifth term of $4a$, $-6ma^2$, $9m^2a^3$.....

2. Find the geometrical mean between $18x^3y$ and $30xy^3z$.

3. Find the ratio when the first and third terms are 5 and 80 respectively.

4. Insert two geometrical means between 8 and 125; and three between 14 and 224.

5. If $a = 2$ and $r = 3$, which term will be equal to 162?

6. The fifth term of a geometrical series is 48, and the ratio 2. Find the first and seventh terms.

7. Find the sum of

 $3 + 6 + 12 + \ldots\ldots$ to eight terms.
 $1 - 3 + 9 - \ldots\ldots$ to seven terms.
 $8 + 4 + 2 + \ldots\ldots$ to ten terms.
 $.1 + .5 + 2.5 + \ldots\ldots$ to seven terms.
 $m - \frac{m}{4} + \frac{m}{16} - \ldots\ldots$ to five terms.

8. The population of a city increases in four years from 10,000 to 14,641. What is the rate of increase?

9. The sum of four numbers in geometrical progression is 200, and the first term is 5. Find the ratio.

10. Find the sum of eight terms of a series whose last term is 1, and fifth term $\frac{1}{8}$.

11. In an odd number of terms, show that the product of the first and last will be equal to the square of the middle term.

12. The product of four terms of a geometrical series is 4, and the fourth term is 4. Determine the series.

13. If from a line one-third be cut off, then one-third of the remainder, and so on, what fraction of the whole will remain when this has been done five times?

14. Of three numbers in geometrical progression, the sum of the first and second exceeds the third by 3, and the sum of the first and third exceeds the second by 21. What are the numbers?

15. Find two numbers whose sum is $3\frac{1}{4}$ and geometrical mean $1\frac{1}{2}$?

16. A glass of wine is taken from a decanter that holds ten glasses, and a glass of water poured in. After this is done five times, what part of the contents is wine?

17. There are four numbers such that the sum of the first and the last is 11, and the sum of the others is 10. The first three of these four numbers are in arithmetical progression, and the last three are in geometrical progression. Find the numbers.

18. Find three numbers in geometrical progression whose sum is 13 and the sum of their squares 91.

19. The difference between two numbers is 48, and the arithmetical mean exceeds the geometrical by 18. Find the numbers.

20. There are four numbers in geometrical progression, the second of which is less than the fourth by 24, and the sum of the extremes is to the sum of the means as 7 to 3. Find the numbers.

21. A number consists of three digits in geometrical progression. The sum of the digits is 13; and if 792 be added to the number, the digits in the units' and hundreds' places will be interchanged. Find the number.

394. When $r < 1$, a geometrical series has its terms continually decreasing; and by increasing n, the value of the nth term, ar^{n-1} may be made as small as we please, though not absolutely zero.

395. The formula for the sum of n terms,

$$\frac{a(1-r^n)}{1-r}$$

may be written $$\frac{a}{1-r} - \frac{ar^n}{1-r}.$$

By increasing n indefinitely, the value of $\frac{ar^n}{1-r}$ becomes indefinitely small, so that the sum of n terms approaches indefinitely to $\frac{a}{1-r}$ as its limit.

Ex. Find the limit of $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} \ldots\ldots$

Here $a = 1$, and $r = -\frac{1}{2}$,

and therefore the limit $\frac{a}{1-r} = \frac{1}{1-(-\frac{1}{2})} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$. *Ans.*

22. Find the limits of the sums of the following infinite series:

$4 + 2 + 1 + \ldots\ldots$	$2 - 1\frac{1}{3} + \frac{8}{9} - \ldots\ldots$
$\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \ldots\ldots$	$.1 + .01 + .001 + \ldots\ldots$
$\frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \ldots\ldots$	$.868686\ldots\ldots$
$1 - \frac{2}{5} + \frac{4}{25} - \ldots\ldots$	$.54444\ldots\ldots$
$\frac{1}{5} + \frac{1}{15} + \frac{1}{45} + \ldots\ldots$	$.83636\ldots\ldots$

*Harmonical Series.

396. A series is called a **Harmonical Series,** or a **Harmonical Progression,** when the *reciprocals* of its terms form an *arithmetical series.*

Hence, the general representative of such a series will be

$$\frac{1}{a}, \; \frac{1}{a+d}, \; \frac{1}{a+2d}, \; \ldots\ldots, \; \frac{1}{a+(n-1)d}.$$

397. Questions relating to harmonical series should be solved by writing the reciprocals of its terms so as to form an arithmetical series.

398. If a and b denote two numbers, and H their harmonical mean, then, by the definition of a harmonical series,

$$\frac{1}{H}-\frac{1}{a}=\frac{1}{b}-\frac{1}{H}.$$

$$\therefore \frac{2}{H}=\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab};$$

$$\therefore H=\frac{2ab}{a+b}.$$

399. Sometimes it is required to insert several harmonical means between two numbers.

Ex. Let it be required to insert three harmonical means between 3 and 18.

Find the three arithmetical means between $\frac{1}{3}$ and $\frac{1}{18}$.

These are found to be $\frac{19}{72}$, $\frac{14}{72}$, $\frac{9}{72}$; therefore, the harmonical means are $\frac{72}{19}$, $\frac{72}{14}$, $\frac{72}{9}$; or $3\frac{15}{19}$, $5\frac{1}{7}$, 8.

* A harmonical series is so called because musical strings of uniform thickness and tension produce *harmony* when their lengths are represented by the *reciprocals* of the natural series of numbers; that is, by the series, 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, etc.

Exercise CXVIII.

1. Insert four harmonical means between 2 and 12.

2. Find two numbers whose difference is 8 and the harmonical mean between them $1\frac{4}{5}$.

3. Find the seventh term of the harmonical series 3, $3\frac{3}{7}$, 4.....

4. Continue to two terms each way the harmonical series two consecutive terms of which are 15, 16.

5. The first two terms of a harmonical series are 5 and 6. Which term will equal 30?

6. The fifth and ninth terms of a harmonical series are 8 and 12. Find the first four terms.

7. The difference between the arithmetical and harmonical means between two numbers is $1\frac{4}{5}$, and one of the numbers is four times the other. Find the numbers.

8. Find the arithmetical, geometrical, and harmonical means between two numbers a and b; and show that the geometrical mean is a mean proportional between the arithmetical and harmonical means. Also, arrange these means in order of magnitude.

9. The arithmetical mean between two numbers exceeds the geometrical by 13, and the geometrical exceeds the harmonical by 12. What are the numbers?

10. The sum of three terms of a harmonical series is 11, and the sum of their squares is 49. Find the numbers.

11. When a, b, c are in harmonical progression, show that $a : c :: a - b : b - c$.

CHAPTER XXII.

BINOMIAL THEOREM.

400. The Binomial Theorem is a formula by means of which a binomial may be raised to any required power without going through the process of multiplication.

WHEN THE EXPONENT IS POSITIVE AND INTEGRAL.

401. Assuming that the laws stated in § 83 for exponents and coefficients hold good for any positive integral exponent n, we have the general formula:

$$(x+a)^n$$
$$= x^n + nx^{n-1}a + \frac{n(n-1)}{1\times 2}x^{n-2}a^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}x^{n-3}a^3 + \cdots\cdots$$

Multiply both members by $x + a$. Then

$$(x+a)^{n+1}$$
$$\begin{aligned} &= x^{n+1} + nx^n a + \frac{n(n-1)}{1\times 2}x^{n-1}a^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}x^{n-2}a^3 + \cdots\cdots \\ &\qquad + x^n a \quad + nx^{n-1}a^2 \qquad\quad + \frac{n(n-1)}{1\times 2}x^{n-2}a^3 + \cdots\cdots \\ \hline &= x^{n+1} + (n+1)x^n a + \frac{(n+1)(n)}{1\times 2}x^{n-1}a^2 + \frac{(n+1)(n)(n-1)}{1\times 2\times 3}x^{n-2}a^3 + \cdots\cdots \end{aligned}$$

The result is in the same *form* as the expansion of $(x+a)^n$, having $n+1$ in place of n.

Therefore, if the theorem is true for a positive integral exponent n, it is true when that exponent is increased by 1.

But in § 83 it was shown to be true for $(x+a)^3$; hence it is true for $(x+a)^4$; and being true for $(x+a)^4$, it is true for $(x+a)^5$; and so on. Hence the theorem is true for any positive integral exponent.

NOTE. A proof of this kind is called *mathematical induction*.

402. If a and x be interchanged, the expansion ~~will~~ proceed by ascending powers of x, as follows:

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\times 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}x^3 + \cdots\cdots$$

403. By comparing the expansion of $(a+x)^n$ with that of $(x+a)^n$, it is obvious that the series in the second members are reversed. Hence the coefficients of any two terms equidistant from the beginning and end of the expansion are equal.

404. If $a=1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \cdots\cdots + nx^{n-1} + x^n.$$

If x be negative, the *odd* powers of x will be *negative*, and the *even* powers *positive*. Thus,

$$(a-x)^n = a^n - na^{n-1}x + \frac{n(n-1)}{1\times 2}a^{n-2}x^2 - \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}x^3 + \cdots\cdots$$

405. To find the *rth* (or general) term in the expansion of $(a+x)^n$.

The following laws will be observed to hold for any term in the expansion of $(a+x)^n$:

1. The exponent of x is less by 1 than the number of the term.

2. The exponent of a is n minus the exponent of x.

3. The last factor of the numerator is greater by 1 than the exponent of a.

4. The last factor of the denominator is the same as the exponent of x.

Therefore, in the rth term:

The exponent of x will be $r-1$.
The exponent of a will be $n-(r-1)$, or $n-r+1$.
The last factor of the numerator will be $n-r+2$.
The last factor of the denominator will be $r-1$.

Hence, the rth term

$$=\frac{n(n-1)(n-2)\cdots\cdots(n-r+2)}{1\times 2\times 3\cdots\cdots(r-1)}a^{n-r+1}x^{r-1}.$$

Thus, the third term of $(a+x)^{20}$ is

$$\frac{20\times 19}{1\times 2}a^{18}x^2=190a^{18}x^2.$$

And the eighth term of $(2x-y)^{11}$ (which is the *fifth* term from the end of the expansion) is

$$\frac{11\times 10\times 9\times 8}{1\times 2\times 3\times 4}(2x)^4(-y)^7=330(16x^4)(-y)^7=-5280x^4y^7.$$

Exercise CXIX.

Expand:

1. $(1+2x)^5$.
2. $(x-3)^8$.
3. $(2x-3y)^4$.
4. $(2-x)^8$.
5. $\left(1-\frac{3y}{4}\right)^5$.
6. $\left(1-\frac{x}{3}\right)^9$.
7. Find the fourth term of $(2x-5y)^{12}$.
8. Find the seventh term of $\left(\frac{x}{2}+\frac{y}{3}\right)^{10}$.
9. Find the twelfth term of $(a^2-ax)^{15}$.
10. Find the eighth term of $(5x^2y-2xy^2)^9$.
11. Find the middle term of $\left(\frac{x}{y}+\frac{y}{x}\right)^8$.

12. Find the middle term of $\left(\frac{x}{y}-\frac{y}{x}\right)^{10}$.

13. Find the two middle terms of $\left(\frac{x}{y}-\frac{y}{x}\right)^{7}$.

14. Find the *rth* term $(2a+x)^n$.

15. Find the *rth* term from the end of $(2a+x)^n$.

16. Find the $(r+4)th$ term of $(a+x)^n$.

17. Find the middle term of $(a+x)^{2m}$.

18. Expand $(2a+x)^{12}$, and find the sum of the terms if $a=1,\ x=-2$.

When the Exponent is Fractional or Negative.

406. The product of

$$1+ax+bx^2+cx^3+\cdots\cdots$$

and

$$1+a'x+b'x^2+c'x^3+\cdots\cdots$$

is an expression in ascending powers of x, as

$$1+Ax+Bx^2+Cx^3+\cdots\cdots,$$

in which the coefficients A, B, C, ... are *functions* of a, b, c, a', b', c', ... ; that is, are made up of these letters in particular ways.

The ways in which a, b, c, a', b', c', ... enter into these functions will evidently be the same, *whatever* may be the *values* assigned to the letters.

Likewise, in the product of

$$1+mx+\frac{m(m-1)}{1\times 2}x^2+\cdots\cdots$$

and

$$1+nx+\frac{n(n-1)}{1\times 2}x^2+\cdots\cdots,$$

the coefficients of x, x^2, ... will be functions of m and n, the *forms* of which will be the same for *all* values of m and n.

But when m and n are positive integers,

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{1 \times 2} x^2 + \cdots\cdots;$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \times 2} x^2 + \cdots\cdots;$$

$$\therefore (1+x)^{m+n} = \text{their product};$$

and $(1+x)^{m+n}$ becomes, by expansion,

$$1 + (m+n)x + \frac{(m+n)(m+n-1)}{1 \times 2} x^2 + \cdots\cdots.$$

These *forms*, then, being true for *all* values of m and n, will hold when m and n are *fractional* or *negative.*

407. If the expressions

$$1 + mx + \frac{m(m-1)}{1 \times 2} x^2 + \cdots\cdots$$

and $$1 + nx + \frac{n(n-1)}{1 \times 2} x^2 + \cdots\cdots$$

be represented by $f(m)$ and $f(n)$, their product will be represented by $f(m+n)$, since it is formed with $m+n$ in the same way as they are formed with m and n.

$$\therefore f(m) \times f(n) = f(m+n).$$

In like manner,

$$f(m) \times f(n) \times f(p) = f(m+n) \times f(p) = f(m+n+p);$$

and so on for any number of such factors

408. For *all* values of m, n, ...

$$f(m) \times f(n) \cdots\cdots \text{ to } s \text{ factors} = f(m+n+\cdots\cdots \text{ to } s \text{ terms}),$$

and if m, n, ... be *all* equal, this equation becomes

$$\{f(m)\}^s = f(ms),$$

in which s is a positive integer.

Now, if ms be any positive integer r, so that m becomes the positive fraction $\frac{r}{s}$, the equation

$$\{f(m)\}^s = f(ms)$$

becomes

$$\left\{f\left(\frac{r}{s}\right)\right\}^s = f(r)$$
$$= (1+x)^r, \text{ since } r \text{ is integral.}$$

$$\therefore (1+x)^{\frac{r}{s}} = f\left(\frac{r}{s}\right)$$
$$= 1 + \frac{r}{s}x + \frac{\frac{r}{s}\left(\frac{r}{s}-1\right)}{1 \times 2}x^2 + \cdots\cdots.$$

Hence, the *form* of the expansion is the same when the exponent is a positive fraction as when it is a positive integer.

409. Again, since the equation

$$f(m) \times f(n) = f(m+n)$$

is true for *all* values of m and n, it is true when $n = -m$.

$$\therefore f(m) \times f(-m) = f(0), \text{ which equals 1.} \quad \S\ 255.$$

$$\therefore f(-m) = \frac{1}{f(m)} = \frac{1}{(1+x)^m} = (1+x)^{-m};$$

that is, $\quad (1+x)^{-m} = f(-m).$

Hence, when the exponent is negative, whether integral or fractional, the *form* of the expansion is the same.

NOTE. In the expansion of $(1+x)^n$, if n is a *positive integer*, the numerator of the last factor of the coefficient of the $(r+1)$*th* term, $n-r+1$, will be equal to 0 when $r = n+1$; this term, therefore, and all following terms (for they will also have this factor) will vanish. Hence, the series will *end* with the *rth* term. But if n is *fractional*, or *negative*, no value of r will make $n-r+1=0$, and the series will be *infinite*. Hence, the sign = in these cases will mean, "*is equal to the limit of the series.*"

Expand to four terms by substituting in the general formula,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1\times 2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}x^3 + \dots\dots :$$

(1) $(1+x)^{\frac{1}{3}}$.

$$(1+x)^{\frac{1}{3}} = 1 + \tfrac{1}{3}x + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1\times 2}x^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{1\times 2\times 3}x^3 + \dots\dots$$

$$= 1 + \tfrac{1}{3}x - \tfrac{1}{9}x^2 + \tfrac{5}{81}x^3 - \dots\dots.$$

(2) $\dfrac{1}{\sqrt[4]{a^2-2ax}}$.

$$\frac{1}{\sqrt[4]{a^2-2ax}} = (a^2-2ax)^{-\frac{1}{4}} = a^{-\frac{1}{2}}\left\{1-\frac{2x}{a}\right\}^{-\frac{1}{4}}$$

$$= \frac{1}{a^{\frac{1}{2}}}\left\{1 + \frac{x}{2a} + \frac{-\frac{1}{4}(-\frac{1}{4}-1)}{1\times 2}\left(\frac{2x}{a}\right)^2 - \frac{-\frac{1}{4}(-\frac{1}{4}-1)(-\frac{1}{4}-2)}{1\times 2\times 3}\left(\frac{2x}{a}\right)^3\right\}$$

$$= \frac{1}{a^{\frac{1}{2}}}\left\{1 + \frac{x}{2a} + \frac{5x^2}{8a^2} + \frac{15x^3}{16a^3} + \dots\dots\right\}.$$

A root may often be extracted by means of an expansion.

(3) Extract the cube root of 344 to six decimal places.

$$344 = 343(1 + \tfrac{1}{343}) = 7^3(1 + \tfrac{1}{343}).$$

$$\therefore \sqrt[3]{344} = 7(1 + \tfrac{1}{343})^{\frac{1}{3}},$$

$$= 7\left(1 + \tfrac{1}{3}\times\tfrac{1}{343} + \frac{\frac{1}{3}(\frac{1}{3}-1)}{1\times 2}(\tfrac{1}{343})^2 + \dots\dots\right),$$

$$= 7(1 + .000971815 - .000000944),$$

$$= 7.006796.$$

(4) Extract the fifth root of 3128 to six decimal places.

$$3128 = 5^5 + 3 = 5^5\left(1 + \frac{3}{5^5}\right).$$

$$\therefore \sqrt[5]{3128} = 5\left(1 + \frac{3}{5^5}\right)^{\frac{1}{5}},$$

$$= 5\left\{1 + \frac{1}{5} \times \frac{3}{5^5} + \frac{\frac{1}{5}(\frac{1}{5} - 1)}{1 \times 2} \times \left(\frac{3}{5^5}\right)^2 + \cdots\right\},$$

$$= 5(1 + .000192 - .000000073728 + \cdots),$$

$$= 5.000959.$$

Exercise CXX.

Expand to four terms:

1. $(1+x)^{\frac{1}{2}}$.
2. $(1+x)^{\frac{2}{3}}$.
3. $(a+x)^{\frac{3}{4}}$.
4. $(1-x)^{-4}$.
5. $(a^2-x^2)^{\frac{5}{2}}$.
6. $(x^2+xy)^{-\frac{3}{2}}$.
7. $(2x-3y)^{-\frac{1}{4}}$
8. $\sqrt[5]{1-5x}$.
9. $\dfrac{1}{\sqrt{(4a^2-3ax)^3}}$.
10. $\sqrt[6]{\dfrac{1}{(1-3y)^5}}$.
11. $(1+x+x^2)^{\frac{2}{3}}$.
12. $(1-x-x^2)^{\frac{3}{2}}$.
13. Find the *rth* term of $(a+x)^{\frac{1}{2}}$.
14. Find the *rth* term of $(a-x)^{-3}$.
15. Find $\sqrt{65}$ to five decimal places.
16. Find $\sqrt[3]{1\frac{1}{30}}$ to five decimal places.
17. Find $\sqrt[7]{129}$ to six decimal places.
18. Expand $(1-2x+3x^2)^{-\frac{2}{5}}$ to four terms.
19. Find the coefficient of x^4 in the expansion of $\dfrac{(1+2x)^2}{(1+3x)^3}$.
20. By means of the expansion of $(1+x)^{\frac{1}{2}}$ show that the limit of the series

$$1 + \frac{1}{2} - \frac{1}{2 \times 2^2} + \frac{1 \times 3}{2 \times 3 \times 2^3} - \frac{1 \times 3 \times 5}{2 \times 3 \times 4 \times 2^4} + \cdots \text{ is } \sqrt{2}.$$

Exercise CXXI.

Apply the formula to the expansion of the following expressions:

1. $(\frac{1}{2}p^5 + 3y^4)^5$.
2. $(\frac{2}{3}a^3 + \frac{3}{4}b^2)^5$.
3. $(a^2b^3 + 2a^3bx^4)^5$.
4. $(\frac{2}{3}ab^2 - \frac{3}{4}a^2y)^7$.
5. $(\sqrt{a} + x)^7$.
6. $(\sqrt{2b} - m)^8$.
7. $(\sqrt{3c} + 2a)^7$.
8. $(\sqrt{\frac{1}{2}a} - 3y)^8$.
9. $(2\sqrt{e} + a)^6$.
10. $(\frac{2}{3}a + \sqrt{2x})^6$.
11. $(\frac{3}{4}a - \sqrt{\frac{1}{2}x})^5$.
12. $(2a - 3\sqrt{y})^6$.
13. $(a^2 + \frac{1}{2}\sqrt{z})^7$.
14. $(\sqrt{b} - \sqrt{y})^8$.
15. $(\sqrt{2c} + \sqrt{3x})^6$.

Find, in the following expansions, the required term:

16. The seventh term of $(2 + a)^{16}$.
17. The eleventh term of $(a + d)^{21}$.
18. The sixth term of $(3 + 2x^2)^9$.
19. The fourteenth term of $(y^3 - 1)^{40}$.
20. The seventh term of $(\frac{1}{2}a - \sqrt{x^3})^{17}$.
21. The fourth term of $(\sqrt{a} - \sqrt[3]{x^2})^{11}$.
22. Develop $(1 + x)^{-1}$, and then make $x = \frac{1}{9}$.
23. Develop $(1 + x)^{\frac{1}{2}}$, and then make $x = \frac{3}{8}$.
24. Develop $(1 + x)^{-\frac{1}{2}}$, and then make $x = 0.003$.

Expand to four terms:

25. $(a + x)^{-5}$.
26. $(a - x)^{-6}$.
27. $(2b - y)^{-6}$.
28. $(\frac{1}{2}c + z)^{-9}$.
29. $(a + \frac{1}{2}x)^{-9}$.
30. $(a^2 + x^3)^{-5}$.

31. $(\sqrt{a} - x^2)^{-6}$. **35.** $(a^2 - 1)^{\frac{1}{2}}$. **39.** $(1 - x^5)^{-\frac{1}{3}}$.

32. $(b + h)^{\frac{1}{2}}$. **36.** $(1 + a)^{\frac{1}{3}}$. **40.** $(1 + 2d)^{-\frac{1}{5}}$.

33. $(b - x)^{\frac{1}{4}}$. **37.** $(a^3 + 1)^{-\frac{1}{2}}$. **41.** $(32 + 5h)^{-\frac{3}{5}}$.

34. $(x^2 + a)^{\frac{1}{2}}$. **38.** $(x^2 - a)^{-\frac{1}{2}}$. **42.** $(9 - 2x^2)^{-\frac{3}{2}}$.

Apply the formula of the binomial to the extraction of the following roots, carrying out the operation to the sixth decimal:

43. $\sqrt{53}$. **45.** $\sqrt[4]{68}$. **47.** $\sqrt[5]{1121}$.

44. $\sqrt[3]{87}$. **46.** $\sqrt[5]{259}$. **48.** $\sqrt[3]{58\frac{1}{8}}$.

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